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ABSTRACT

On a plane earth the ionospheric or tropospheric range error depends only on the total refractivity content or zeroth moment of the refracting layer and the elevation angle. On a spherical earth, however, the dependence is more complex; so for more accurate results it has been necessary to resort to complex ray-tracing calculations.

This report offers a simple, high-accuracy alternative to the ray-tracing calculation. By appropriate expansion of the angular dependence in the ray-tracing integral in a power series in height, an expression is obtained for the range error in the form

$$\Delta R = \sum_{m=0}^{N} G_m(E)\, M_m$$

where $G_m(E)$ is a simple function of elevation angle at the expansion height

M_m is the m^{th} moment of the refractivity distribution about the expansion height, h_c.

The rapidity of convergence is heavily dependent on the choice of expansion height. For expansion heights in the neighborhood of the centroid of the layer (300-490 km), the expansion to N = 2 (three terms) gives results accurate to about 0.4% at $E = 10^{o}$. This is considered quite good in view of the simplicity of the formulation. As an analytic tool, the expansion affords some valuable insight on the influence of layer shape on range errors in special problems.

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MOMENT EXPANSION FOR IONOSPHERIC RANGE ERROR

1. INTRODUCTION

At a given frequency, the ionospheric range error correction to a zero order approximation valid for plane earth is directly proportional to the total electron columnar content or zeroth order moment of the electron density profile and is a simple function of the elevation angle, csc E. For a spherical earth, a first order correction is available which depends only on the effective height or first order moment of the electron density profile.

These considerations lead to the possibility of an expansion of the range error correction in terms of successive moments of the profile distribution in which the coefficients are elevation angle functions. If such a formulation is capable of providing sufficient accuracy in a reasonably small number of terms, it would have significant computational advantages over straightforward ray tracing for the case where corrections at a large number of points (i.e., elevation angles) must be computed for the same ionosphere.

Freeman (Reference 1) made a start in this direction but found relatively slow convergence. His expansion was in terms of moments about the ground level, h = 0. It can be reasoned that the expansion would necessarily be more rapidly convergent if moments were taken about a point in the ionosphere such as the centroid. For example, it is clear, to the extent that the ionospheric density profile can be approximated by a thin shell at its centroid, that a single term in the moment expansion (the zeroth moment) would give an exact answer for the case of moments about the centroid where all higher order moments are already zero, but not for moments about any other point.

2. STRAIGHT PATH ASSUMPTION

The expansion of the range error correction is based on neglecting the effect of bending on the range error by assuming that the ray path is a straight line. The range of validity of this assumption can be developed as follows.

A derivation given in Appendix A shows that, for an idealized uniform slab ionosphere and a plane earth, the straight line group and the bent path group range errors are related by equation (A27) as follows:

$$\Delta R_{S_g} - \Delta R_{B_g} = -\frac{3N^2}{2} r \left(1 - \frac{r}{h}\right) \frac{\sin^2\Phi}{\cos^3\Phi} \qquad (1)$$

or from equation (A28)

$$\frac{\Delta R_{B_g}}{\Delta R_{S_g}} = 1 - \frac{3N}{2}\left(1 - \frac{r}{h}\right)\tan^2 \Phi \qquad (2)$$

where (refer to Appendix A)

ΔR_{S_g} = straight line group range error

$= \rho_{S_g} - R$

ΔR_{B_g} = bent path group range error

$= \rho_{B_g} - R$

N = refractivity (phase) = $n - 1 < 0$

r = ionosphere layer thickness

h = height of satellite

Φ = true geometrical zenith angle

R = geometrical range

where in terms of the refractivity integrals (Reference 6)

$$\rho_{S_g} = \frac{d}{df}\left[f\,\rho_{S_p}\right]$$

$$\Delta R_{S_g} = \frac{d}{df}\left[f \int_{\substack{\text{straight}\\\text{line}}} N\,ds\right] = \int_{\substack{\text{straight}\\\text{line}}} \frac{d}{df}(fN)\,ds \qquad (3)$$

or $$\Delta R_{S_g} = \int_{\substack{\text{straight}\\\text{line}}} (\text{group refractivity})\,ds$$

Similarly define

$$\rho_{B_g} = \frac{d}{df}\left[f\,\rho_{B_p}\right] \qquad (4)$$

$$\Delta R_{B_g} = \frac{d}{df}\left[f \int_{\substack{\text{bent}\\\text{phase}\\\text{ray}\\\text{path}}} N\,ds\right]$$

It should be observed, however, that the bent path group range error is defined in terms of the derivative of the phase measured range error and is not simply the integral of the group refractivity along the bent path (refer to Reference 2 for a thorough discussion):

$$\rho_{B_g} \neq \int_{\substack{\text{bent}\\\text{path}}} \frac{d}{df}(fN)\, ds$$

or

$$\rho_{B_g} \neq \int_{\substack{\text{bent}\\\text{path}}} (\text{group refractivity})\, ds$$

To illustrate with a numerical example, consider the case where

r = 293.57 km

h = 1333 km

f = 434 MHz (Secor effective two-way frequency)

f_o = 5.679 MHz = vertical incidence critical frequency

so (Reference 5)

$$N = -\frac{1}{2}\left(\frac{f_o}{f}\right)^2$$

$$= -84.84 \times 10^{-6}$$

Then as a function of Φ, we have from equation (2)

Φ	$\Delta R_{B_g} / \Delta R_{S_g}$	
0	1	← (vertical ray - no bending)
45	1.000099	
70	1.000749	

There is no need to consider Φ larger than 70^o since, in application to the spherical earth problem, Φ should be taken as the local zenith angle in the ionosphere and this cannot significantly exceed 70^o for a ray emanating from the surface of the earth. Neglecting bending thus results in a worst case error of about 0.07% in this idealization.

A numerical comparison of the straight and bent path integrals for a spherical earth was carried out (Reference 3) for a Chapman ionosphere with a maximum refractivity

($N_{max} = -84.84 \times 10^{-6}$ at $h_{max} = 375$ km) and the satellite at 1333 km. The straight path integral was carried out by a single ray trace program which calculates $\sum_{i=1}^{n} N_i \Delta s_i$, and the bent path integral was carried out by the REEK ray trace program modified for group range errors as discussed in Reference 2. The resulting differences between the outputs of these two programs are presented in Table 1. The greatest difference, 0.11 meter out of 73 meters total refraction or 0.15%, is in reasonable agreement with the simple theory of the difference at low elevation angles, equation (1), but is dominated at the higher elevation angles by a bias of about 0.04 meter, presumably due to the difference in the REEK and straight path integration formulas. These results are taken as confirming the approximate equations (1) and (2) and the adequacy of the straight path assumption for the problem at hand.

Table 1. Difference Between Straight Path Group Ray Trace (ΔR_{S_g}) and Modified REEK Group Ray Trace (ΔR_{B_g})

Elevation (degrees) ($E_o = 90 - \Phi_o$)	ΔR_{B_g} (meters)	$\Delta R_{S_g} - \Delta R_{B_g}$ (meters)	
		Numerical	Theoretical (Eq 1)
0.1	73.336	- 0.1140	- 0.0621
1	73.245	- 0.1122	- 0.0618
10	65.812	- 0.0715	- 0.0432
20	52.960	- 0.0424	- 0.0199
40	35.911	- 0.0355	- 0.0039
60	28.311	- 0.0395	- 0.0008
80	25.360	- 0.0428	- 0.000069
89	25.027	- 0.0434	- 0.0000006

The above is based on a spherical earth and a Chapman profile where

$$N_{max} = -84.84 \times 10^{-6}$$

$$h_{max} = 375 \text{ km} = \text{height of } N_{max}$$

$$H_s = 108 \text{ km} = \text{scale height}$$

$$h_{sat} = 1333 \text{ km}$$

The figures in the theoretical column are based on equation (1) corrected for local angle of incidence at 375 km and using $r = 108$ (e) = 293.57 km. $\simeq$ effective thickness.

3. MOMENT EXPANSION DERIVATION

Based on the straight path assumption, the ionospheric range error for a plane earth is computed simply as

$$\Delta R = \int_{\substack{\text{straight} \\ \text{path}}} (n - 1)\, ds = \int_{o}^{h_s} \sec \Phi_o\, N(h)\, dh$$

on a spherical earth this becomes

$$\Delta R = \int_{o}^{h_s} \sec\,(\Phi(h))\, N(h)\, dh \tag{5}$$

where $N(h) = n(h) - 1$

$\Phi(h)$ is the angle the straight line ray makes with the local vertical as it passes through height h.

From the spherical geometry shown in Figure 1, this is given by

$$(a + h) \sin \Phi(h) = a \sin \Phi_o \tag{6}$$

where Φ_o = zenith angle at surface of earth = $90 - E_o$

h = height above earth

a = radius of earth.

thus

$$\sin \Phi(h) = \frac{a}{a + h} \sin \Phi_o = \frac{a}{a + h} \cos E_o$$

$$\sec \Phi(h) = \left[1 - \left(\frac{a}{a + h} \cos E_o\right)^2\right]^{-1/2} \tag{7}$$

Now, following the motivation discussed in the introduction, we expand sec (Φ (h)) in a Taylor series in h about some arbitrary reference height, h_c, which will normally be chosen somewhere near the center of the layer, thus

$$\sec(\Phi(h)) = \sec(\Phi(h)) + (h - h_c) \frac{d}{dh} \sec(\Phi(h))$$

$$+ \ldots + \frac{(h - h_c)^m}{m!} \frac{d^m}{dh^m} (\sec \Phi(h)) \tag{8}$$

where the derivitives are to be evaluated at $h = h_c$

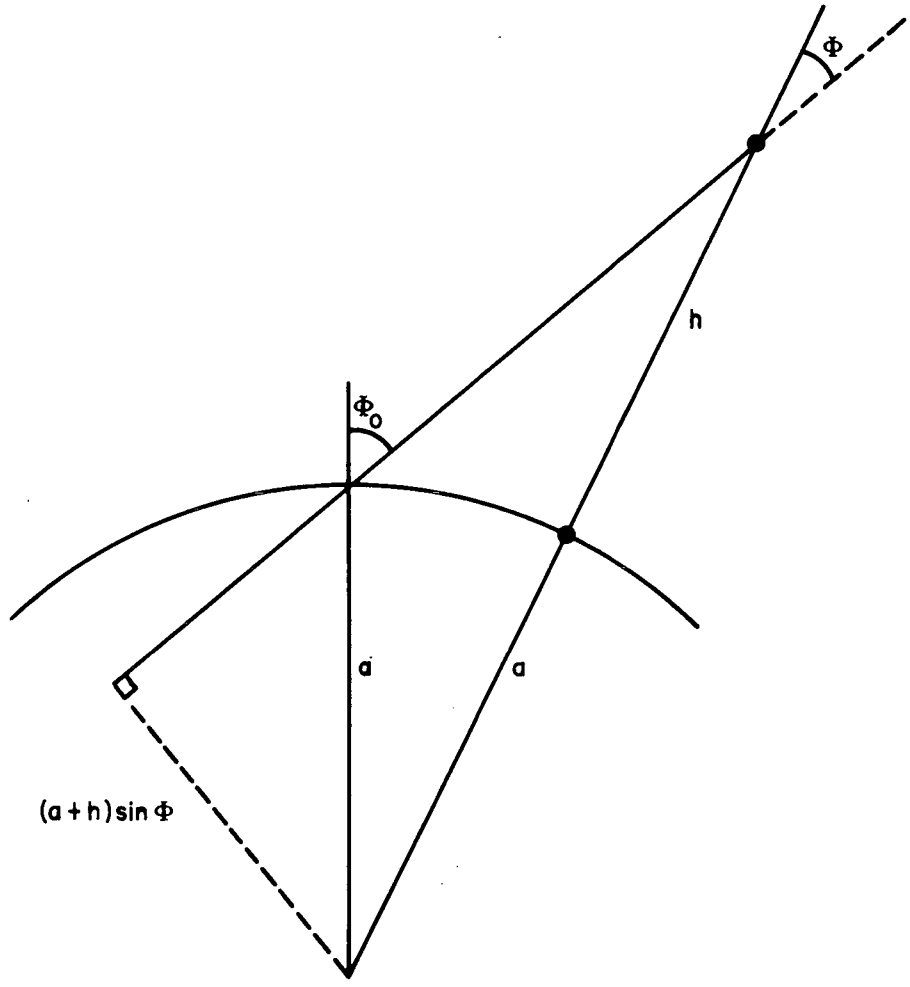

Figure 1. Geometry of the Straight Line Path Angle

thus

$$\Delta R = \int_0^{h_s} \left(\sum_{m=0}^{\infty} G_m (h - h_c)^m \right) N(h)\, dh \tag{9}$$

where

$$G_m = \frac{1}{m!} \frac{d^m}{dh^m} \sec(\Phi(h)) \Big|_{h = h_c} \tag{10}$$

Note that the geometrical factor $G_m(E)$ is independent of h. This means that when we carry out the integration in equation (9), interchanging integral and summation, the geometrical factors come out of the integrals and we have

$$\Delta R = \sum_{m=0}^{\infty} G_m(E_o) M_m \tag{11}$$

where

$$M_m = \int (h - h_c)^m N(h)\, dh \tag{12}$$

$$= \text{mth moment about } h_c \text{ of refractivity distribution}$$

By defining

$$C = (a \cos E_o)^2$$

$$r = a + h_c ,$$

the first few geometrical coefficients may be written out as

$$G_o(E_o) = \sec \Phi_c = \left(1 - \left(\frac{a \cos E_o}{r}\right)^2\right)^{-1/2} = \left(1 - \frac{C}{r^2}\right)^{-1/2}$$

$$G_1(E_o) = -CG_o^3/r^3$$

$$G_2(E_o) = (3/2)\, CG_o^5/r^4$$

$$G_3(E_o) = (1/2)\, CG_o^5 (-5G_o^2 + 1)/r^5$$

$$G_4(E_o) = (5/8)\, CG_o^7 (7G_o^2 - 3)/r^6$$

$$G_5(E_o) = (3/8)\, CG_o^7 (-21G_o^4 + 14G_o^2 - 1)/r^7$$

$$G_6(E_o) = (7/16)\, CG_o^9 (33G_o^4 - 30G_o^2 + 5)/r^8 \tag{13}$$

For the first or m = 0 term of this series, equation (11) becomes

$$\Delta R = \left[1 - \left(\frac{a}{a + h_c} \cos E_o\right)^2\right]^{-1/2} \int_o^{h_s} N \, dh$$

which is recognized as a pretty good zeroth order approximation and is used in the GSFC programs DC, DODS, NAP3 (Reference 4). Higher order terms introduce the higher order angle dependence as a function of the higher order moments of the refractivity distribution in height. Note that if h_c is chosen as the centroid of the distribution, the first order moment, M_1, vanishes by definition.

The general nth order partial summation of equation (11) to be used for practical computation is

$$\Delta R_N = \sum_{m=0}^{N} G_m(E_o) M_m \tag{14}$$

4. CHAPMAN LAYER TESTS

In order to gain some notion as to the practical rate of convergence of the moment series, equation (14), it is necessary to compare partial sums of the series to some reference, such as a ray trace of the same profile. The REEK ray trace program was used for this reference.

The first series of comparisons was made using a modified Chapman profile of ionization density or refractivity for which

$$N(h) = N_{max} \exp(1 - z - \exp(-z))$$

where

$$z = \frac{h - h_m}{H_s}$$

h_m = height of maximum refractivity = 375 km

H_s = scale height = 108 km

N_{max} = maximum refractivity = -2×10^{-4}

The moment expansion was compared to the REEK reference for elevation angles from 0^o to 89^o as a function of the moment expansion center, h_c, in equation (12) and for various orders of expansion N in equation (14). The results of this

comparison are shown in Tables B-1 through B-11. Table B-1 gives the actual correction ΔR as computed by equation (14) for orders from 0 through 6. This is for expansion about 437 km which happens to be the centroid of the profile. For comparison, the last column in the table lists the REEK result. Tables B-2 through B-11 give the difference between the expansion of order N and REEK, denoted DIF N, for various elevation angles. The bottom two rows of each table give the average and rms errors over the chosen set of elevation angles. The various tables are for expansion centers, h_c, varying from 300 to 480 km.

It is of particular interest to examine the rate of convergence of the series with increasing order N as a function of the expansion center, h_c. It will be immediately noted that the rate of convergence for low elevation angles ($E < 10^o$) is very poor for h_c less than the centroid height of 437 km. For the higher order expansions (i. e., N = 6), the convergence is seemingly better as the expansion height increases, at least up to h_c = 480 km, the maximum tested.

For a simple low order expansion, however, the most attractive strategy seems to be expansion to the second order about the centroid (refer to Table 1). It should be noted that this result is as good as the expansion to the sixth order about 480 km. Specifically, the maximum error is only 0.09 meter (at $E = 10^o$) out of 155 meters total correction, or 0.058%. Further, since the first order moment about the centroid vanishes, there are only two terms (m = 0 and m = 2) in the expansion for this case, making it particularly simple to compute. Of course, the first order moment is implicit in finding the centroid.

5. REAL LAYER TESTS

For the second series of comparisons, actual measured electron density profiles were utilized. As a basis for this comparison, 16 ionization density profiles were available from the ionospheric soundings taken during the GEOS-2 Geodetic Systems Intercomparison Experiment conducted by Goddard at Wallops Island during April through June of 1968. There are composite profiles based on nearly simultaneous topside (Alouette I and II) and bottomside (Wallops Station) ionograms. Reduction of the ionograms and preparation of the composite profiles were performed by Mr. John Jackson of the Goddard Space Flight Center (GSFC). Based on the preceding results, the centroid was used as the center of expansion for these cases which, of course, differed for each profile.

The frequency used was 434.2696 MHz, the effective Sequential Collation of Range (Secor) frequency. The moments of the refractivity profiles and reference ray trace (REEK) results are given in Appendix C. Moments about zero height can be converted to moments about any other height h_c (e.g., the centroid) by the relation

$$M_n(h_c) = \sum_{j=0}^{n} \binom{n}{j} \left(-h_c\right)^j M_{n-j}(o)$$

The results of the moment expansions are listed in Tables D-1 through D-16 where the bottom two rows of each table give the average and rms errors over the chosen set of elevation angles. Typical convergence behavior versus order of summation is shown in Figure 2 for $E = 10^o$. It will be noted from the detailed tabulations that the error at 10^o is reasonably in accord with the mean and rms error over the set of elevation angles in each case.

Both the convergence behavior plot of Figure 2 and the detailed tabulations again confirm that for expansion about the layer centroid, the second order summation (N = 2, two terms) is a good practical choice of order and of the same order of accuracy as the sixth order summation, N = 6.

Accuracy of the N = 0 (using M_o only) and N = 2 (using M_o, M_1, M_2) expansions at $E = 10^o$ on the 16 WICE profiles is summarized in Tables 2 and 3. At vertical incidence, the N = 0 (i.e., M_o only) expansion is of course perfect. At 10^o, however, the M_o term alone introduces an rms error of 2.64m or 3.1%. Inclusion of the M_1 and M_2 terms reduces this to 0.338m or 0.4% which is considered excellent agreement in view of the simplicity of the formulation.

The rms error (relative to REEK) over all 16 days of the N = 0 and N = 2 expansions for the real profile data set is plotted against the elevation angle in Figure 3. Note that for 0.4-meter accuracy at 434 MHz, the N = 0 or zeroth order approximation is accurate for elevation angles down to about 27^o while the N = 2 expansion extends this down to 1^o.

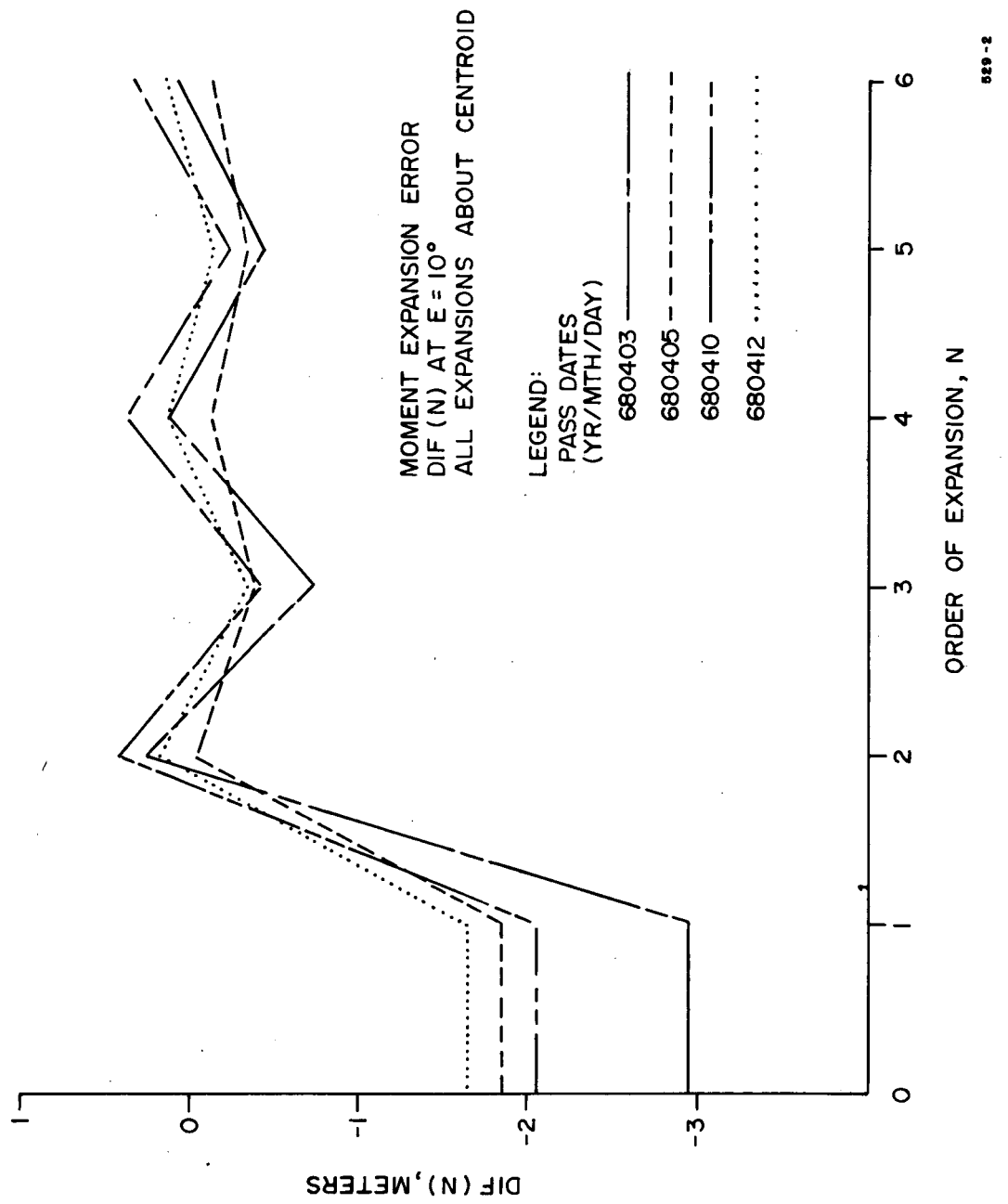

Figure 2. Moment Expansion Error DIF (N) at E = 10^{o}

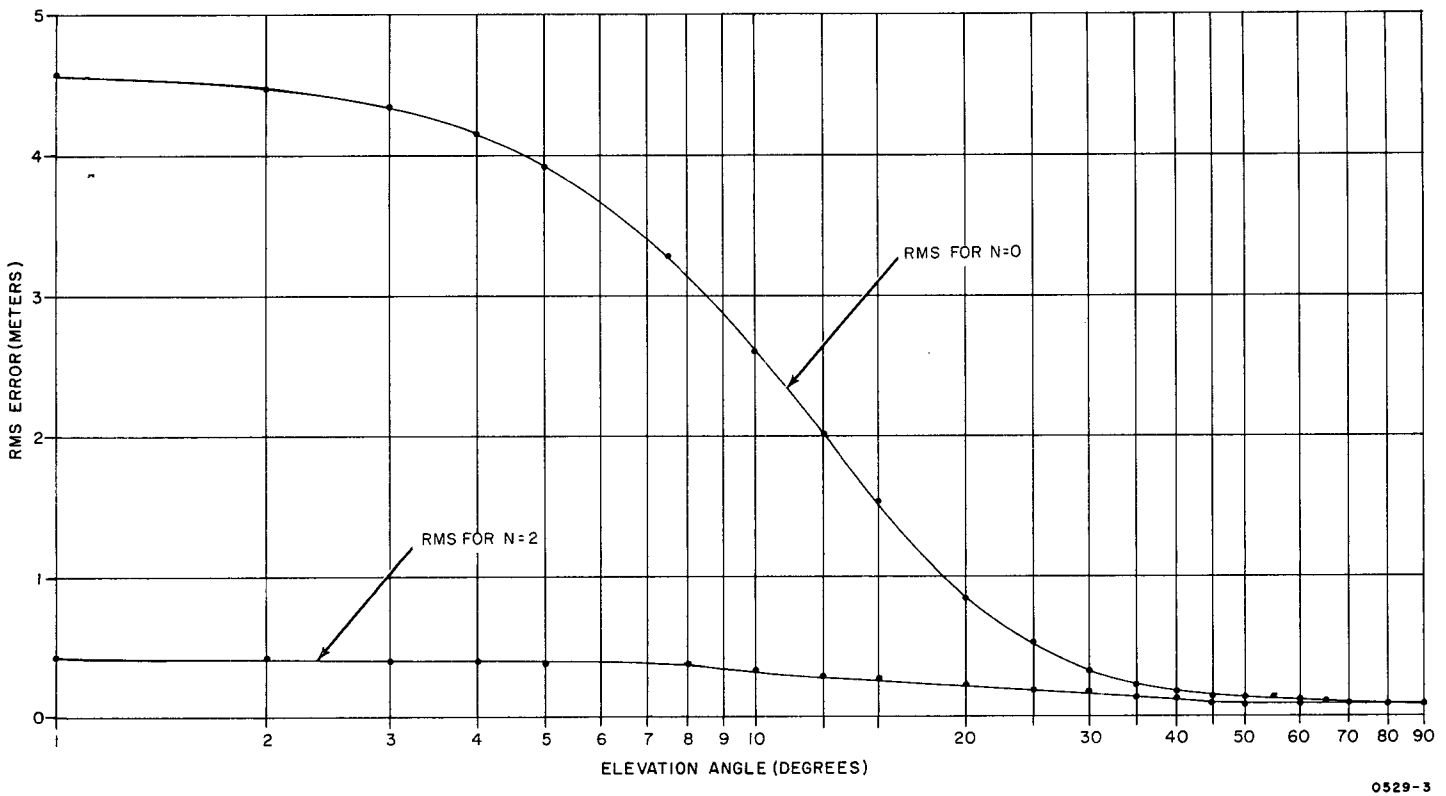

Figure 3. RMS Error of the N=0 and N=2 Expansion versus Elevation Angle

Table 2. Summary of Moment Expansion Comparisons with Ray Trace Measured Profiles. Expansion to Zero Order.

Date YYMMDD	ΔR @ $E = 10^{o}$ (m)	DIF (0) @ $E = 10^{o}$ (m)	DIF (0) AVG (m)	DIF (0) RMS (m)
680403	114.98	-2.950	-1.741	2.613
680405	77.12	-1.817	-1.080	1.563
680410	89.84	-2.045	-1.198	1.879
680412	75.60	-1.664	-0.968	1.471
680417	86.05	-2.010	-1.178	1.807
680524	119.17	-3.383	-2.080	2.796
680525	71.73	-1.729	-1.033	1.496
680529	85.01	-1.720	-1.007	1.476
680530	75.00	-1.670	-0.973	1.472
680604	71.93	-1.465	-0.860	1.273
680605	93.47	-1.988	-1.162	1.783
680611	75.66	-3.679	-2.268	3.610
680613	84.76	-4.141	-2.544	3.973
680618	118.06	-5.052	-3.107	4.883
680621	52.32	-0.961	-0.555	0.898
680625	77.43	-1.655	-0.979	1.483
Mean	85.50	-2.370	-1.421	2.155
RMS	87.57	2.642	1.600	2.445
Standard deviation	18.36	1.132	0.713	1.119

DIF (0) = (Moment expansion to order zero) - (ray trace)

Table 3. Summary of Moment Expansion Comparisons with Ray Trace Measured Profiles. Expansion to second order

Date YYMMDD	ΔR @ E = 10^{o} (m)	DIF 2 @ E = 10^{o} (m)	DIF 2 AVG (m)	DIF 2 RMS (m)
680403	114.98	0.257	0.117	0.242
680405	77.12	-0.044	-0.065	0.076
680410	89.84	0.414	0.247	0.352
680412	75.60	0.197	0.105	0.169
680417	86.05	0.289	0.159	0.246
680524	119.17	-0.833	-0.635	0.676
680525	71.73	-0.003	-0.037	0.082
680529	85.01	0.040	-0.003	0.071
680530	75.00	0.163	0.083	0.143
680604	71.93	0.067	0.021	0.081
680605	93.47	0.248	0.138	0.212
680611	75.66	0.411	0.164	0.204
680613	84.76	0.193	0.001	0.123
680618	118.06	0.564	0.225	0.307
680621	52.32	0.293	0.185	0.249
680625	77.43	0.251	0.139	0.242
Mean	85.50	0.157	0.052	0.217
RMS	87.57	0.338	0.206	0.261
Standard deviation	18.36	0.308	0.205	0.149

DIF 2 = (Moment expansion to order 2) - (ray trace)

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APPENDIX A. ANALYSIS OF STRAIGHT PATH ASSUMPTION

APPENDIX A. ANALYSIS OF STRAIGHT PATH ASSUMPTION

For a number of purposes, it is convenient, and often quite accurate, to ignore the bending of the path and to compute range error as an integral of refractivity along the straight line or geometrical path. As a guide to the application of this assumption, it is useful to have a simple first order understanding of the error involved in the case of a simplified model for which an analytic solution is possible.

This simplified model is shown in Figure A-1 and corresponds to a simple slab of uniform ionosphere extending from h_1 to $h_1 + r$. The ground point is at the origin and the satellite at x, h. Angles are defined on the figure.

We are concerned with the case of small refraction, i. e. $N << 1$, and will seek simple approximations valid in this case. In the approximations, we will use the nomenclature O_j to denote neglect of terms of order j and higher in the refractivity $N = n - 1$.

From Snell's law

$$\begin{aligned} \sin \Phi_1 &= n \sin \Phi_2 \\ &= n \sin (\Phi_1 + \beta) \\ &\approx n (\sin \Phi_1 + \beta \cos \Phi_1) + O_2 \end{aligned} \tag{A1}$$

or

$$\begin{aligned} \beta &\approx -(n - 1) \tan \Phi_1 + O_2 \\ &\approx -N \tan \Phi + O_2 \end{aligned} \tag{A2}$$

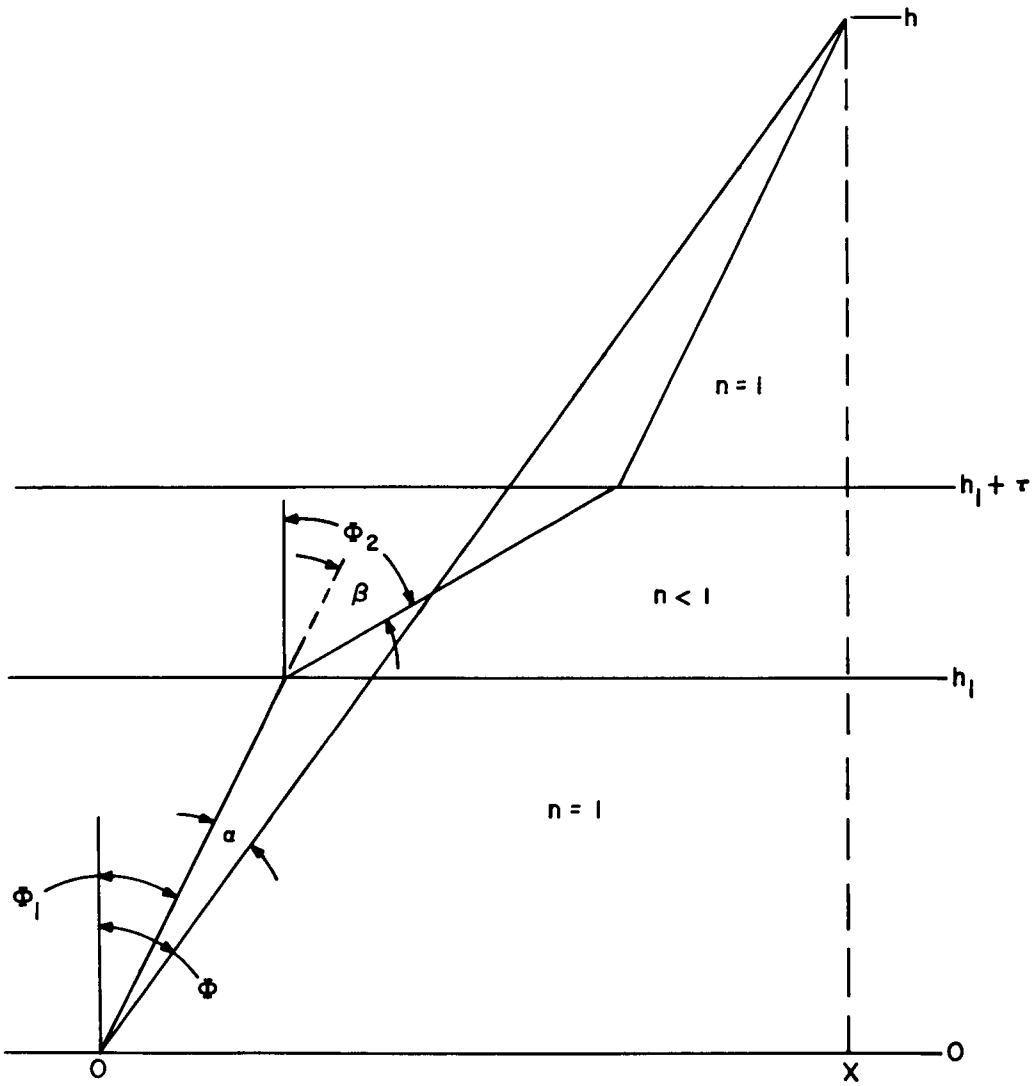
Also, since both the bent and geometrical rays are to start and end up in the same place, we can write, carrying terms to second order,

$$x = h \tan \Phi = (h - r) \tan \Phi_1 + r \tan \Phi_2$$

Using $\Phi_1 = \Phi - \alpha$ and $\Phi_2 = \Phi - (\alpha - \beta)$ and the fact that α and β are small, we can expand in a Taylor series to get

$$\operatorname{Tan} \Phi_1 \approx \operatorname{Tan} \Phi - \alpha \frac{d \operatorname{Tan} \Phi}{d \Phi} + \frac{\alpha^2}{2} \frac{d^2 \operatorname{Tan} \Phi}{d \Phi^2} + O_3$$

$$\operatorname{Tan} \Phi_2 \approx \operatorname{Tan} \Phi - (\alpha - \beta) \frac{d \operatorname{Tan} \Phi}{d \Phi} + \frac{(\alpha - \beta)^2}{2} \frac{d^2 \operatorname{Tan} \Phi}{d \Phi^2} + O_3$$



$\Phi_1 = \Phi - \alpha$

$\Phi_2 = \Phi_1 + \beta$

$\Phi_2 = \Phi - \alpha + \beta$

529-A-1

Figure A-1. Idealized Ionosphere Geometry for Straight Line and Bent Paths

then

$$x = (h - r)(\tan\Phi - \alpha \sec^2 \Phi + \alpha^2 \sec^2 \Phi \tan \Phi)$$

$$+ r\left[\tan \Phi - (\alpha - \beta) \sec^2 \Phi + (\alpha - \beta)^2 \sec^2 \Phi \tan \Phi\right] + O_3$$

$$x = h\tan \Phi - (h\alpha - r\beta) \sec^2 \Phi + (h\alpha^2 - 2r\alpha\beta + r\beta^2) \sec^2 \Phi \tan \Phi + O_3 \quad \text{(A3)}$$

or

$$(h\alpha - r\beta) = (h\alpha^2 - 2r\alpha\beta + r\beta^2) \tan \Phi + O_3 \quad \text{(A4)}$$

whence, to the first order

$$h\alpha = r\beta + O_2 \quad \text{(A5)}$$

and from equation (A2)

$$\alpha = -\frac{r}{h} N \tan \Phi + O_2 \quad \text{(A6)}$$

so

$$\alpha - \beta = \left(1 - \frac{r}{h}\right) N \tan \Phi + O_2$$

Now we can write four different ways of computing phase range.

1. Geometrically along the straight path:

$$R_S = \int_S 1 \, ds = h \sec \Phi \quad \text{(A7)}$$

2. Refractive index along the straight path:

$$\rho_{S_p} = \int_S n \, ds$$

$$= \left[(h - r) + n\ r\right] \sec \Phi$$

$$= \left[h + N r\right] \sec \Phi \qquad \text{(A8)}$$

$$= R_S + N r \sec \Phi$$

where

$$N = n - 1$$

3. Geometrically along the bent path:

$$R_{B_p} = \int_{B_p} 1\, ds$$

$$= (h - r) \sec \Phi_1 + r \sec \Phi_2 \qquad \text{(A9)}$$

It will be clear that the difference between R_B and R_S must be of second order in the angles α and β, so the expansions must be consistent to this order. Accordingly, using the Taylor series expansion for $\sec(\Phi - \alpha)$ and $\sec\left[\Phi - (\alpha - \beta)\right]$ we get

$$R_{B_p} = (h - r)\left[\sec \Phi - \alpha \sec \Phi \tan \Phi + \frac{\alpha^2}{2} \sec \Phi (\tan^2 \Phi + \sec^2 \Phi)\right]$$

$$+ r\left[\sec \Phi - (\alpha - \beta) \sec \Phi \tan \Phi + \frac{(\alpha - \beta)^2}{2} \sec \Phi (\tan^2 \Phi + \sec^2 \Phi)\right] + O_3$$

$$= h \sec \Phi - (h\alpha - r\beta) \sec \Phi \tan \Phi$$

$$+ \frac{(h\alpha^2 - 2 r\alpha\beta + r\beta^2)}{2} \sec \Phi (\tan^2 \Phi + \sec^2 \Phi) + O_3 \qquad \text{(A10)}$$

Then, using equation (A4)

$$R_{B_p} = h \sec \Phi - (h\alpha^2 - 2 r \alpha\beta + r\beta^2) \sec \Phi \tan^2 \Phi$$

$$+ \left(\frac{h\alpha^2 - 2 r \alpha\beta + r\beta^2}{2}\right) \sec \Phi (2 \tan^2 \Phi + 1) + O_3 \qquad \text{(A11)}$$

$$R_{B_p} = h \sec \Phi + \frac{h\alpha^2 - 2 r\alpha\beta + r\beta^2}{2} \sec \Phi + O_3 \qquad \text{(A11)}$$

Now, using equation (A5)

$$R_{B_p} = h \sec \Phi - \frac{r\beta}{2} (\alpha - \beta) \sec \Phi + O_3 \tag{A12}$$

Then, using equation (A2) for β and equation (A6) for $\alpha - \beta$, we get

$$R_{B_p} = h \sec \Phi + \frac{N^2}{2} r \left(1 - \frac{r}{h}\right) \sec \Phi \tan^2 \Phi + O_3$$

4. Refractivity along the bent path:

$$\rho_{B_p} = \int_{B_p} n \, ds \tag{A15}$$

$$= (h - r) \sec \Phi_1 + (N + 1) \, r \sec \Phi_2$$

$$= R_{B_p} + N r \sec \Phi_2 \tag{A16}$$

$$= R_{B_p} + N r \left[\sec \Phi - (\alpha - \beta) \sec \Phi \tan \Phi \right.$$

$$\left. + \frac{(\alpha - \beta)^2}{2} \sec \Phi \, (\tan^2 \Phi + \sec^2 \Phi) + O_3 \right] \tag{A17}$$

Using equation (A6) for $\alpha - \beta$

$$\rho_{B_p} = R_{B_p} + N r \sec \Phi - N^2 r \left(1 - \frac{r}{h}\right) \sec \Phi \tan^2 \Phi + O_3 \tag{A18}$$

$$= h \sec \Phi + N r \sec \Phi - \frac{N^2}{2} r \left(1 - \frac{r}{h}\right) \sec \Phi \tan^2 \Phi + O_3 \tag{A19}$$

These four cases are written together here to stress the differences:

$$R_S = \frac{h}{\cos \Phi} \tag{A20}$$

$$\rho_{S_p} = \frac{h}{\cos \Phi} + \frac{N r}{\cos \Phi} \tag{A21}$$

$$R_{B_p} \approx \frac{h}{\cos \Phi} + \frac{N^2 r}{2} \left(1 - \frac{r}{h}\right) \frac{\tan^2 \Phi}{\cos \Phi} \tag{A22}$$

$$\rho_{B_p} \approx \frac{h}{\cos \Phi} + \frac{N r}{\cos \Phi} - \frac{N^2 r}{2} \left(1 - \frac{r}{h}\right) \frac{\tan^2 \Phi}{\cos \Phi} \tag{A23}$$

where R denotes geometrical length

ρ denotes apparent radio range

S denotes straight path

B denotes bent path

p denotes phase related

The above (except R) all relate to phase quantities, i. e., phase path and phase range. From the phase range, however, we can immediately derive the group range (denoted by subscript g) in terms of terminal characteristics by (Reference 6)

$$\rho_{B_g} = \frac{d}{df} \left(f \rho_{B_p} \right)$$

and using

$$N = - \frac{K}{f^2}$$

$$\rho_{B_g} = \frac{h}{\cos \Phi} - \frac{N r}{\cos \Phi} + \frac{3N^2 r}{2} \left(1 - \frac{r}{h}\right) \frac{\tan^2 \Phi}{\cos \Phi} \tag{A24}$$

Similarly

$$\rho_{S_g} = \frac{d}{df} \left(f \rho_{S_p} \right) = \frac{h}{\cos \Phi} - \frac{N r}{\cos \Phi} \tag{A26}$$

so that

$$\rho_{S_g} - \rho_{B_g} = - \frac{3}{2} N^2 r \left(1 - \frac{r}{h}\right) \frac{\tan^2 \Phi}{\cos \Phi} \tag{A27}$$

or in terms of errors

$$\Delta R_{B_g} = \rho_{B_g} - R_S = - \frac{N r}{\cos \Phi} + \frac{3N^2 r}{2} \left(1 - \frac{r}{h}\right) \frac{\tan^2 \Phi}{\cos \Phi}$$

$$\Delta R_{S_g} = \rho_{S_g} - R_S = - \frac{N r}{\cos \Phi}$$

$$\frac{\Delta R_{B_g}}{\Delta R_{S_g}} = 1 - \frac{3}{2} N \left(1 - \frac{r}{h}\right) \tan^2 \Phi \tag{A28}$$

APPENDIX B. MOMENT EXPANSION AND REEK COMPARISON—CHAPMAN REFRACTIVITY PROFILE

APPENDIX B. MOMENT EXPANSION AND REEK COMPARISON-CHAPMAN REFRACTIVITY PROFILE

The tables in this section have been compiled based on the following parameters.

$N_{max} = 200 \times 10^{-6}$

$h_{max} = 375$

$H_S = 108$ km

$h_{sat} = 1333$ km

$h_{centroid} = 437$ km

Table B-1. Ionospheric Range Error Correction, ΔR (N), from Moment Expansion to Order N About 437 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	ΔR (N) (meters)							REEK Ray Trace
		$\Delta R0$	$\Delta R1$	$\Delta R2$	$\Delta R3$	$\Delta R4$	$\Delta R5$	$\Delta R6$	
0.00	437	167.07	167.07	172.93	171.23	173.40	171.50	173.87	172.92
1.00	437	166.89	166.89	172.72	171.03	173.18	171.30	173.64	172.70
2.00	437	166.36	166.36	172.09	170.44	172.53	170.71	172.95	172.06
3.00	437	165.48	165.48	171.05	169.46	171.45	169.74	171.83	171.01
4.00	437	164.28	164.28	169.64	168.13	169.99	168.42	170.31	169.58
5.00	437	162.77	162.77	167.88	166.47	168.17	166.76	168.42	167.81
7.50	437	157.86	157.86	162.20	161.07	162.35	161.36	162.46	162.11
10.00	437	151.72	151.72	155.23	154.39	155.27	154.64	155.28	155.14
12.50	437	144.86	144.86	147.59	147.00	147.56	147.20	147.54	147.52
15.00	437	137.69	137.69	139.77	139.37	139.71	139.51	139.68	139.72
20.00	437	123.68	123.68	124.83	124.65	124.78	124.72	124.76	124.82
25.00	437	111.16	111.16	111.78	111.70	111.75	111.73	111.74	111.79
30.00	437	100.52	100.52	100.87	100.83	100.85	100.84	100.84	100.88
35.00	437	91.70	91.70	91.90	91.88	91.89	91.89	91.89	91.92
40.00	437	84.46	84.46	84.58	84.57	84.57	84.57	84.57	84.59
45.00	437	78.54	78.54	78.61	78.60	78.61	78.61	78.61	78.62
50.00	437	73.71	73.71	73.75	73.75	73.75	73.75	73.75	73.77
55.00	437	69.79	69.79	69.82	69.82	69.82	69.82	69.82	69.83
60.00	437	66.63	66.63	66.64	66.64	66.64	66.64	66.64	66.66
65.00	437	64.11	64.11	64.12	64.12	64.12	64.12	64.12	64.13
70.00	437	62.16	62.16	62.16	62.16	62.16	62.16	62.16	62.17
75.00	437	60.69	60.69	60.70	60.70	60.70	60.70	60.70	60.70
80.00	437	59.68	59.68	59.68	59.68	59.68	59.68	59.68	59.69
89.00	437	58.89	58.89	58.89	58.89	58.89	58.89	58.89	58.90

Table B-2. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 300 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	300	25.775	-16.677	13.369	-16.164	22.297	-38.614	73.561
1.00	300	25.683	-16.556	13.245	-15.953	21.949	-37.883	71.950
2.00	300	25.394	-16.214	12.867	-15.354	20.929	-35.799	67.339
3.00	300	24.920	-15.671	12.264	-14.421	19.348	-32.620	60.379
4.00	300	24.278	-14.956	11.477	-13.233	17.365	-28.710	51.968
5.00	300	23.489	-14.105	10.557	-11.880	15.157	-24.461	43.044
7.50	300	21.045	-11.637	8.016	-8.345	9.691	-14.482	23.189
10.00	300	18.252	-9.108	5.638	-5.338	5.474	-7.474	10.555
12.50	300	15.453	-6.872	3.757	-3.210	2.830	-3.535	4.262
15.00	300	12.869	-5.068	2.418	-1.868	1.375	-1.606	1.579
20.00	300	8.691	-2.681	0.953	-0.625	0.278	-0.349	0.158
25.00	300	5.805	-1.424	0.368	-0.228	0.033	-0.106	-0.020
30.00	300	3.890	-0.778	0.141	-0.098	-0.017	-0.051	-0.034
35.00	300	2.628	-0.441	0.051	-0.052	-0.024	-0.033	-0.029
40.00	300	1.788	-0.259	0.015	-0.032	-0.022	-0.024	-0.024
45.00	300	1.220	-0.158	-0.000	-0.023	-0.019	-0.020	-0.019
50.00	300	0.829	-0.099	-0.006	-0.018	-0.016	-0.016	-0.016
55.00	300	0.557	-0.064	-0.009	-0.015	-0.014	-0.014	-0.014
60.00	300	0.365	-0.042	-0.009	-0.013	-0.012	-0.012	-0.012
65.00	300	0.229	-0.029	-0.010	-0.011	-0.011	-0.011	-0.011
70.00	300	0.133	-0.020	-0.009	-0.010	-0.010	-0.010	-0.010
75.00	300	0.067	-0.015	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	300	0.024	-0.011	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	300	-0.008	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		10.141	-5.537	3.961	-4.455	5.690	-9.411	16.990
RMS		14.504	8.623	6.552	10.755	10.117	17.048	31.617

Table B-3. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 325 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	325	18.512	-12.198	8.338	-9.618	11.784	-19.100	32.781
1.00	325	18.454	-12.112	8.269	-9.499	11.616	-18.763	32.117
2.00	325	18.270	-11.874	8.051	-9.166	11.114	-17.806	30.202
3.00	325	17.964	-11.496	7.701	-8.648	10.330	-16.339	27.288
4.00	325	17.546	-10.998	7.241	-7.984	9.336	-14.518	23.727
5.00	325	17.029	-10.403	6.698	-7.222	8.219	-12.517	19.898
7.50	325	15.402	-8.665	5.173	-5.198	5.391	-7.695	11.130
10.00	325	13.503	-6.859	3.708	-3.423	3.130	-4.157	5.275
12.50	325	11.557	-5.237	2.517	-2.124	1.656	-2.069	2.204
15.00	325	9.723	-3.908	1.647	-1.276	0.815	-0.995	0.828
20.00	325	6.679	-2.110	0.663	-0.456	0.157	-0.249	0.064
25.00	325	4.518	-1.139	0.257	-0.179	0.005	-0.088	-0.033
30.00	325	3.056	-0.631	0.096	-0.083	-0.024	-0.047	-0.036
35.00	325	2.078	-0.361	0.032	-0.046	-0.025	-0.032	-0.030
40.00	325	1.420	-0.215	0.006	-0.030	-0.022	-0.024	-0.024
45.00	325	0.972	-0.132	-0.004	-0.022	-0.019	-0.020	-0.019
50.00	325	0.662	-0.084	-0.008	-0.017	-0.016	-0.016	-0.016
55.00	325	0.445	-0.055	-0.010	-0.014	-0.014	-0.014	-0.014
60.00	325	0.291	-0.037	-0.010	-0.013	-0.012	-0.012	-0.012
65.00	325	0.182	-0.026	-0.010	-0.011	-0.011	-0.011	-0.011
70.00	325	0.105	-0.018	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	325	0.052	-0.014	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	325	0.017	-0.011	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	325	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		7.435	-4.108	2.513	-2.711	3.057	-4.771	7.720
RMS		10.543	6.354	4.130	6.472	5.405	8.540	14.258

Table B-4. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 350 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	350	12.059	-9.245	5.121	-5.913	6.306	-9.819	15.046
1.00	350	12.031	-9.181	5.084	-5.842	6.224	-9.656	14.763
2.00	350	11.932	-9.006	4.962	-5.649	5.973	-9.197	13.938
3.00	350	11.765	-8.729	4.763	-5.350	5.576	-8.491	12.673
4.00	350	11.534	-8.365	4.498	-4.965	5.070	-7.610	11.112
5.00	350	11.243	-7.930	4.183	-4.521	4.494	-6.633	9.414
7.50	350	10.306	-6.651	3.283	-3.325	3.008	-4.224	5.425
10.00	350	9.171	-5.310	2.395	-2.248	1.782	-2.385	2.648
12.50	350	7.967	-4.092	1.653	-1.437	0.955	-1.251	1.126
15.00	350	6.796	-3.081	1.097	-0.891	0.470	-0.639	0.416
20.00	350	4.777	-1.692	0.449	-0.341	0.077	-0.186	0.008
25.00	350	3.286	-0.927	0.173	-0.143	-0.013	-0.077	-0.041
30.00	350	2.250	-0.519	0.061	-0.071	-0.028	-0.045	-0.037
35.00	350	1.543	-0.301	0.016	-0.042	-0.027	-0.032	-0.030
40.00	350	1.060	-0.181	-0.001	-0.029	-0.023	-0.024	-0.024
45.00	350	0.729	-0.112	-0.008	-0.021	-0.019	-0.019	-0.019
50.00	350	0.497	-0.072	-0.010	-0.017	-0.016	-0.016	-0.016
55.00	350	0.335	-0.048	-0.011	-0.014	-0.014	-0.014	-0.014
60.00	350	0.219	-0.033	-0.011	-0.012	-0.012	-0.012	-0.012
65.00	350	0.136	-0.023	-0.010	-0.011	-0.011	-0.011	-0.011
70.00	350	0.078	-0.017	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	350	0.038	-0.013	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	350	0.011	-0.011	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	350	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		4.990	-3.148	1.569	-1.703	1.656	-2.516	3.597
RMS		6.993	4.842	2.564	4.019	2.919	4.443	6.612

Table B-5. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 375 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	375	6.280	-7.394	2.995	-3.791	3.362	-5.260	7.026
1.00	375	6.276	-7.343	2.978	-3.746	3.323	-5.175	6.904
2.00	375	6.249	-7.205	2.915	-3.628	3.198	-4.945	6.540
3.00	375	6.199	-6.988	2.809	-3.445	2.997	-4.591	5.977
4.00	375	6.124	-6.702	2.666	-3.211	2.738	-4.148	5.275
5.00	375	6.025	-6.361	2.494	-2.941	2.440	-3.652	4.503
7.50	375	5.675	-5.357	1.990	-2.204	1.657	-2.406	2.649
10.00	375	5.200	-4.300	1.478	-1.527	0.992	-1.421	1.312
12.50	375	4.645	-3.334	1.037	-1.004	0.531	-0.788	0.552
15.00	375	4.064	-2.526	0.697	-0.642	0.253	-0.430	0.187
20.00	375	2.975	-1.404	0.288	-0.262	0.025	-0.146	-0.026
25.00	375	2.106	-0.778	0.107	-0.119	-0.026	-0.069	-0.046
30.00	375	1.471	-0.440	0.033	-0.063	-0.032	-0.043	-0.038
35.00	375	1.022	-0.257	0.004	-0.039	-0.028	-0.031	-0.030
40.00	375	0.709	-0.156	-0.007	-0.027	-0.023	-0.024	-0.024
45.00	375	0.490	-0.098	-0.011	-0.021	-0.019	-0.019	-0.019
50.00	375	0.336	-0.064	-0.012	-0.017	-0.016	-0.016	-0.016
55.00	375	0.226	-0.043	-0.012	-0.014	-0.014	-0.014	-0.014
60.00	375	0.147	-0.030	-0.011	-0.012	-0.012	-0.012	-0.012
65.00	375	0.091	-0.021	-0.010	-0.011	-0.011	-0.011	-0.011
70.00	375	0.051	-0.016	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	375	0.023	-0.012	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	375	0.005	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	375	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		2.766	-2.535	0.933	-1.115	0.887	-1.385	1.694
RMS		3.791	3.885	1.519	2.600	1.569	2.408	3.114

Table B-6. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 400 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	400	1.066	-6.350	1.525	-2.577	1.730	-2.960	3.275
1.00	400	1.083	-6.305	1.522	-2.546	1.714	-2.913	3.224
2.00	400	1.117	-6.187	1.497	-2.468	1.654	-2.791	3.063
3.00	400	1.166	-6.002	1.452	-2.349	1.556	-2.605	2.809
4.00	400	1.225	-5.758	1.388	-2.196	1.426	-2.372	2.489
5.00	400	1.289	-5.467	1.309	-2.020	1.276	-2.109	2.134
7.50	400	1.443	-4.612	1.070	-1.537	0.872	-1.437	1.265
10.00	400	1.542	-3.711	0.813	-1.088	0.520	-0.890	0.619
12.50	400	1.561	-2.885	0.581	-0.733	0.270	-0.523	0.242
15.00	400	1.507	-2.193	0.396	-0.482	0.116	-0.307	0.058
20.00	400	1.264	-1.228	0.162	-0.210	-0.010	-0.122	-0.046
25.00	400	0.974	-0.685	0.055	-0.102	-0.035	-0.064	-0.049
30.00	400	0.717	-0.390	0.011	-0.057	-0.034	-0.042	-0.039
35.00	400	0.516	-0.230	-0.006	-0.037	-0.028	-0.031	-0.030
40.00	400	0.367	-0.140	-0.012	-0.027	-0.023	-0.024	-0.024
45.00	400	0.257	-0.089	-0.013	-0.021	-0.019	-0.019	-0.019
50.00	400	0.177	-0.058	-0.013	-0.017	-0.016	-0.016	-0.016
55.00	400	0.119	-0.039	-0.012	-0.014	-0.014	-0.014	-0.014
60.00	400	0.077	-0.028	-0.011	-0.012	-0.012	-0.012	-0.012
65.00	400	0.046	-0.020	-0.011	-0.011	-0.011	-0.011	-0.011
70.00	400	0.024	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	400	0.009	-0.012	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	400	-0.001	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	400	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		0.731	-2.184	0.486	-0.773	0.454	-0.804	0.787
RMS		0.932	3.341	0.790	1.780	0.813	1.372	1.461

Table B-7. Ionosphere Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 437 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	437	-5.853	-5.853	0.008	-1.698	0.478	-1.429	0.947
1.00	437	-5.811	-5.811	0.017	-1.675	0.479	-1.405	0.936
2.00	437	-5.702	-5.702	0.029	-1.625	0.466	-1.350	0.892
3.00	437	-5.531	-5.531	0.042	-1.549	0.440	-1.268	0.818
4.00	437	-5.306	-5.306	0.055	-1.452	0.405	-1.167	0.724
5.00	437	-5.038	-5.038	0.068	-1.341	0.362	-1.052	0.616
7.50	437	-4.251	-4.251	0.089	-1.035	0.240	-0.754	0.347
10.00	437	-3.421	-3.421	0.090	-0.748	0.128	-0.501	0.140
12.50	437	-2.662	-2.662	0.075	-0.519	0.045	-0.322	0.018
15.00	437	-2.026	-2.026	0.054	-0.352	-0.006	-0.209	-0.040
20.00	437	-1.137	-1.137	0.014	-0.166	-0.043	-0.101	-0.063
25.00	437	-0.636	-0.636	-0.008	-0.087	-0.044	-0.060	-0.052
30.00	437	-0.364	-0.364	-0.017	-0.052	-0.037	-0.041	-0.039
35.00	437	-0.215	-0.215	-0.019	-0.035	-0.029	-0.031	-0.030
40.00	437	-0.132	-0.132	-0.018	-0.026	-0.023	-0.024	-0.024
45.00	437	-0.084	-0.084	-0.016	-0.020	-0.019	-0.019	-0.019
50.00	437	-0.055	-0.055	-0.015	-0.017	-0.016	-0.016	-0.016
55.00	437	-0.038	-0.038	-0.013	-0.014	-0.014	-0.014	-0.014
60.00	437	-0.027	-0.027	-0.012	-0.012	-0.012	-0.012	-0.012
65.00	437	-0.019	-0.019	-0.011	-0.011	-0.011	-0.011	-0.011
70.00	437	-0.015	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	437	-0.012	-0.012	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	437	-0.010	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	437	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		-2.015	-2.015	0.016	-0.520	0.115	-0.409	0.212
RMS		3.079	3.079	0.040	1.180	0.228	0.674	0.423

Table B-8. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 450 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	450	-7.987	-5.900	-0.398	-1.555	0.226	-1.171	0.569
1.00	450	-7.938	-5.858	-0.386	-1.535	0.229	-1.151	0.564
2.00	450	-7.807	-5.748	-0.365	-1.488	0.225	-1.107	0.538
3.00	450	-7.600	-5.575	-0.338	-1.418	0.214	-1.042	0.492
4.00	450	-7.327	-5.349	-0.305	-1.330	0.196	-0.961	0.432
5.00	450	-6.999	-5.079	-0.269	-1.229	0.174	-0.870	0.364
7.50	450	-6.024	-4.285	-0.181	-0.951	0.107	-0.634	0.189
10.00	450	-4.977	-3.449	-0.112	-0.690	0.043	-0.430	0.055
12.50	450	-3.995	-2.684	-0.068	-0.481	-0.004	-0.285	-0.023
15.00	450	-3.148	-2.043	-0.045	-0.329	-0.033	-0.190	-0.058
20.00	450	-1.909	-1.146	-0.030	-0.158	-0.051	-0.096	-0.066
25.00	450	-1.159	-0.641	-0.028	-0.084	-0.046	-0.059	-0.052
30.00	450	-0.717	-0.366	-0.026	-0.051	-0.037	-0.041	-0.039
35.00	450	-0.455	-0.216	-0.023	-0.035	-0.029	-0.031	-0.030
40.00	450	-0.296	-0.132	-0.020	-0.026	-0.024	-0.024	-0.024
45.00	450	-0.196	-0.084	-0.017	-0.020	-0.019	-0.019	-0.019
50.00	450	-0.132	-0.055	-0.015	-0.017	-0.016	-0.016	-0.016
55.00	450	-0.090	-0.038	-0.013	-0.014	-0.014	-0.014	-0.014
60.00	450	-0.061	-0.027	-0.012	-0.012	-0.012	-0.012	-0.012
65.00	450	-0.041	-0.020	-0.011	-0.011	-0.011	-0.011	-0.011
70.00	450	-0.028	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	450	-0.019	-0.012	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	450	-0.013	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	450	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		-2.872	-2.031	-0.112	-0.478	0.045	-0.342	0.117
RMS		4.286	3.104	0.180	1.082	0.110	0.556	0.254

Table B-9. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 460 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	460	-9.613	-5.997	-0.697	-1.481	0.061	-1.022	0.348
1.00	460	-9.558	-5.955	-0.683	-1.461	0.067	-1.004	0.347
2.00	460	-9.410	-5.843	-0.655	-1.416	0.068	-0.966	0.330
3.00	460	-9.177	-5.668	-0.617	-1.350	0.066	-0.910	0.301
4.00	460	-8.867	-5.438	-0.570	-1.266	0.060	-0.841	0.261
5.00	460	-8.495	-5.164	-0.517	-1.170	0.051	-0.764	0.214
7.50	460	-7.381	-4.358	-0.381	-0.907	0.020	-0.562	0.095
10.00	460	-6.170	-3.509	-0.262	-0.659	-0.013	-0.388	0.004
12.50	460	-5.020	-2.731	-0.176	-0.461	-0.038	-0.262	-0.049
15.00	460	-4.014	-2.078	-0.119	-0.317	-0.052	-0.178	-0.070
20.00	460	-2.508	-1.166	-0.063	-0.153	-0.057	-0.094	-0.068
25.00	460	-1.565	-0.652	-0.042	-0.082	-0.048	-0.058	-0.053
30.00	460	-0.993	-0.373	-0.032	-0.050	-0.038	-0.041	-0.039
35.00	460	-0.643	-0.220	-0.026	-0.034	-0.030	-0.031	-0.030
40.00	460	-0.425	-0.135	-0.021	-0.026	-0.024	-0.024	-0.024
45.00	460	-0.285	-0.085	-0.018	-0.020	-0.019	-0.019	-0.019
50.00	460	-0.193	-0.056	-0.015	-0.017	-0.016	-0.016	-0.016
55.00	460	-0.131	-0.038	-0.013	-0.014	-0.014	-0.014	-0.014
60.00	460	-0.088	-0.027	-0.012	-0.012	-0.012	-0.012	-0.012
65.00	460	-0.059	-0.020	-0.011	-0.011	-0.011	-0.011	-0.011
70.00	460	-0.038	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	460	-0.024	-0.012	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	460	-0.015	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	460	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		-3.528	-2.065	-0.207	-0.456	-0.001	-0.302	0.061
RMS		5.211	3.156	0.331	1.030	0.040	0.487	0.156

Table B-10. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 470 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	470	-11.185	-6.142	-0.980	-1.434	-0.079	-0.909	0.177
1.00	470	-11.124	-6.099	-0.964	-1.415	-0.072	-0.893	0.179
2.00	470	-10.961	-5.985	-0.930	-1.371	-0.065	-0.859	0.170
3.00	470	-10.702	-5.806	-0.882	-1.307	-0.060	-0.810	0.152
4.00	470	-10.359	-5.571	-0.822	-1.226	-0.057	-0.750	0.127
5.00	470	-9.944	-5.291	-0.754	-1.133	-0.055	-0.683	0.098
7.50	470	-8.697	-4.467	-0.571	-0.879	-0.056	-0.508	0.021
10.00	470	-7.331	-3.598	-0.407	-0.640	-0.062	-0.356	-0.037
12.50	470	-6.019	-2.801	-0.280	-0.448	-0.067	-0.244	-0.069
15.00	470	-4.860	-2.132	-0.191	-0.309	-0.069	-0.169	-0.079
20.00	470	-3.095	-1.197	-0.096	-0.150	-0.062	-0.091	-0.070
25.00	470	-1.966	-0.669	-0.057	-0.081	-0.049	-0.058	-0.053
30.00	470	-1.266	-0.382	-0.039	-0.050	-0.038	-0.041	-0.040
35.00	470	-0.830	-0.225	-0.029	-0.034	-0.030	-0.031	-0.030
40.00	470	-0.553	-0.138	-0.023	-0.025	-0.024	-0.024	-0.024
45.00	470	-0.373	-0.087	-0.019	-0.020	-0.019	-0.019	-0.019
50.00	470	-0.253	-0.057	-0.016	-0.017	-0.016	-0.016	-0.016
55.00	470	-0.171	-0.039	-0.014	-0.014	-0.014	-0.014	-0.014
60.00	470	-0.115	-0.027	-0.012	-0.012	-0.012	-0.012	-0.012
65.00	470	-0.076	-0.020	-0.011	-0.011	-0.011	-0.011	-0.011
70.00	470	-0.049	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	470	-0.030	-0.012	-0.009	-0.010	-0.010	-0.010	-0.010
80.00	470	-0.018	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	470	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		-4.166	-2.116	-0.297	-0.442	-0.040	-0.272	0.017
RMS		6.108	3.233	0.474	0.998	0.047	0.435	0.083

Table B-11. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About 480 km and REEK Ray Trace of a Chapman Ionosphere

Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)						
		N0	N1	N2	N3	N4	N5	N6
0.00	480	-12.705	-6.329	-1.252	-1.412	-0.200	-0.825	0.043
1.00	480	-12.640	-6.285	-1.233	-1.392	-0.192	-0.810	0.047
2.00	480	-12.462	-6.168	-1.194	-1.349	-0.182	-0.779	0.045
3.00	480	-12.179	-5.984	-1.136	-1.286	-0.170	-0.736	0.036
4.00	480	-11.803	-5.743	-1.064	-1.207	-0.159	-0.682	0.023
5.00	480	-11.348	-5.456	-0.981	-1.115	-0.147	-0.623	0.006
7.50	480	-9.974	-4.609	-0.756	-0.865	-0.123	-0.467	-0.037
10.00	480	-8.459	-3.714	-0.547	-0.630	-0.105	-0.331	-0.070
12.50	480	-6.993	-2.893	-0.381	-0.442	-0.093	-0.230	-0.086
15.00	480	-5.686	-2.203	-0.262	-0.305	-0.084	-0.162	-0.087
20.00	480	-3.671	-1.237	-0.129	-0.149	-0.066	-0.090	-0.072
25.00	480	-2.360	-0.692	-0.072	-0.081	-0.051	-0.058	-0.053
30.00	480	-1.535	-0.395	-0.046	-0.050	-0.039	-0.041	-0.040
35.00	480	-1.014	-0.233	-0.032	-0.034	-0.030	-0.030	-0.030
40.00	480	-0.679	-0.142	-0.025	-0.025	-0.024	-0.024	-0.024
45.00	480	-0.460	-0.090	-0.020	-0.020	-0.019	-0.019	-0.019
50.00	480	-0.313	-0.059	-0.016	-0.017	-0.016	-0.016	-0.016
55.00	480	-0.212	-0.040	-0.014	-0.014	-0.014	-0.014	-0.014
60.00	480	-0.142	-0.028	-0.012	-0.012	-0.012	-0.012	-0.012
65.00	480	-0.093	-0.020	-0.011	-0.011	-0.011	-0.011	-0.011
70.00	480	-0.059	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010
75.00	480	-0.035	-0.012	-0.010	-0.010	-0.010	-0.010	-0.010
80.00	480	-0.020	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009
89.00	480	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009
AVG		-4.786	-2.182	-0.384	-0.436	-0.074	-0.250	-0.017
RMS		6.978	3.333	0.613	0.982	0.100	0.396	0.042

APPENDIX C. MOMENT AND RAY TRACE DATA OF THE 16 WICE COMPOSITE REFRACTIVITY PROFILES

Table C-1. Moments of the WICE Composite Refractivity Profiles and REEK Reference Ray Trace Results

Date (1968)	h_c	$M_m = \int (h - h_c)^m N(h)\, dh$ = m^{th} moment of refractivity distribution (h_c = centroid) (meters^{m+1})						
Mo-Day	k_m	M_0	M_1	M_2	M_3	M_4	M_5	M_6
4-03	433	0.4332450 + 2	0.1069145 - 7	0.5043384 + 12	0.3321553 + 17	0.4327780 + 22	0.3416097 + 27	0.2684275 + 32
4-05	460	0.2975695 + 2	0.9199932 - 8	0.3155736 + 12	0.1756411 + 17	0.2347594 + 22	0.1666015 + 27	0.1271815 + 32
4-10	407	0.3321381 + 2	0.1056377 - 7	0.3412645 + 12	0.2446158 + 17	0.3160283 + 22	0.2669140 + 27	0.2168146 + 32
4-12	444	0.2886210 + 2	0.8099077 - 8	0.3087713 + 12	0.1914888 + 17	0.2475425 + 22	0.1884881 + 27	0.1452903 + 32
4-17	426	0.3231524 + 2	0.8235773 - 8	0.3499091 + 12	0.2361816 + 17	0.3102253 + 22	0.2511005 + 27	0.2005333 + 32
5-24	483	0.4658194 + 2	0.9114931 - 8	0.5032127 + 12	0.2443654 + 17	0.3364399 + 22	0.2180486 + 27	0.1620946 + 32
5-25	442	0.2726176 + 2	0.4059600 - 8	0.2827326 + 12	0.1760657 + 17	0.2239714 + 22	0.1709978 + 27	0.1313573 + 32
5-29	467	0.3311481 + 2	0.5917489 - 8	0.3242890 + 12	0.1866457 + 17	0.2340581 + 22	0.1674715 + 27	0.1241645 + 32
5-30	445	0.2864385 + 2	0.1021752 - 7	0.3053398 + 12	0.1860048 + 17	0.2384129 + 22	0.1790702 + 27	0.1370259 + 32
6-04	449	0.2760212 + 2	0.8728088 - 8	0.2593139 + 12	0.1553437 + 17	0.1926867 + 22	0.1441983 + 27	0.1087082 + 32
6-05	427	0.3519597 + 2	0.8796436 - 8	0.3413403 + 12	0.2090037 + 17	0.2637599 + 22	0.2040193 + 27	0.1579325 + 32
6-11	389	0.2678067 + 2	0.8720802 - 8	0.5166261 + 12	0.3324208 + 17	0.5602314 + 22	0.4536000 + 27	0.4092231 + 32
6-13	409	0.3053767 + 2	0.8976153 - 8	0.6058212 + 12	0.3428873 + 17	0.6153685 + 22	0.4526691 + 27	0.4087450 + 32
6-18	392	0.4215764 + 2	0.1497136 - 7	0.7201949 + 12	0.4781466 + 17	0.7688350 + 22	0.6263306 + 27	0.5551166 + 32
6-21	400	0.1930349 + 2	0.7902012 - 8	0.1676601 + 12	0.1210371 + 17	0.1512410 + 22	0.1290964 + 27	0.1047821 + 32
6-25	410	0.2873375 + 2	0.1156852 - 7	0.2680322 + 12	0.1986926 + 17	0.2457947 + 22	0.2094402 + 27	0.1682737 + 32

APPENDIX D. MOMENT EXPANSION AND REEK COMPARISON — REAL DATA REFRACTIVITY PROFILE

(Spacecraft height 1,000,000 meters — Expansion center = centroid)

Table D-1. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 433 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680403	0.00	433	-4.928	-4.928	0.450	-1.610	0.567	-1.222	0.565	128.41
680403	1.00	433	-4.896	-4.896	0.452	-1.592	0.562	-1.204	0.557	128.24
680403	2.00	433	-4.809	-4.809	0.448	-1.548	0.542	-1.160	0.525	127.76
680403	3.00	433	-4.672	-4.672	0.439	-1.481	0.507	-1.094	0.474	126.96
680403	4.00	433	-4.491	-4.491	0.424	-1.394	0.460	-1.011	0.408	125.88
680403	5.00	433	-4.274	-4.274	0.405	-1.293	0.405	-0.916	0.334	124.54
680403	7.50	433	-3.632	-3.632	0.338	-1.014	0.255	-0.671	0.151	120.24
680403	10.00	433	-2.950	-2.950	0.257	-0.748	0.120	-0.463	0.013	114.98
680403	12.50	433	-2.319	-2.319	0.177	-0.534	0.023	-0.317	-0.065	109.24
680403	15.00	433	-1.786	-1.786	0.106	-0.379	-0.037	-0.224	-0.100	103.38
680403	20.00	433	-1.034	-1.034	0.010	-0.204	-0.084	-0.136	-0.109	92.23
680403	25.00	433	-0.605	-0.605	-0.037	-0.130	-0.089	-0.103	-0.097	82.52
680403	30.00	433	-0.369	-0.369	-0.056	-0.098	-0.083	-0.087	-0.085	74.42
680403	35.00	433	-0.239	-0.239	-0.062	-0.081	-0.076	-0.077	-0.077	67.77
680403	40.00	433	-0.165	-0.165	-0.062	-0.072	-0.069	-0.070	-0.070	62.35
680403	45.00	433	-0.121	-0.121	-0.061	-0.065	-0.064	-0.065	-0.064	57.94
680403	50.00	433	-0.095	-0.095	-0.058	-0.061	-0.060	-0.060	-0.060	54.35
680403	55.00	433	-0.078	-0.078	-0.056	-0.057	-0.057	-0.057	-0.057	51.44
680403	60.00	433	-0.067	-0.067	-0.054	-0.055	-0.054	-0.054	-0.054	49.10
680403	65.00	433	-0.060	-0.060	-0.052	-0.052	-0.052	-0.052	-0.052	47.24
680403	70.00	433	-0.055	-0.055	-0.051	-0.051	-0.051	-0.051	-0.051	45.79
680403	75.00	433	-0.052	-0.052	-0.050	-0.050	-0.050	-0.050	-0.050	44.71
680403	80.00	433	-0.050	-0.050	-0.049	-0.049	-0.049	-0.049	-0.049	43.96
680403	89.00	433	-0.048	-0.048	-0.048	-0.048	-0.048	-0.048	-0.048	43.38
680403	AVG		-1.741	-1.741	0.117	-0.528	0.105	-0.385	0.081	
680403	RMS		2.613	2.613	0.242	1.135	0.266	0.586	0.251	

Table D-2. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 460 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680405	0.00	460	-2.903	-2.903	-0.008	-0.891	0.011	-0.617	-0.043	85.46
680405	1.00	460	-2.886	-2.886	-0.007	-0.883	0.010	-0.611	-0.044	85.36
680405	2.00	460	-2.840	-2.840	-0.006	-0.864	0.004	-0.595	-0.052	85.06
680405	3.00	460	-2.766	-2.766	-0.007	-0.833	-0.005	-0.571	-0.063	84.57
680405	4.00	460	-2.668	-2.668	-0.009	-0.794	-0.019	-0.541	-0.079	83.91
680405	5.00	460	-2.550	-2.550	-0.012	-0.748	-0.034	-0.506	-0.096	83.08
680405	7.50	460	-2.197	-2.197	-0.025	-0.618	-0.076	-0.413	-0.138	80.41
680405	10.00	460	-1.817	-1.817	-0.044	-0.491	-0.114	-0.331	-0.167	77.12
680405	12.50	460	-1.460	-1.460	-0.065	-0.386	-0.139	-0.268	-0.179	73.49
680405	15.00	460	-1.153	-1.153	-0.083	-0.306	-0.151	-0.224	-0.179	69.74
680405	20.00	460	-0.710	-0.710	-0.108	-0.209	-0.153	-0.174	-0.163	62.53
680405	25.00	460	-0.449	-0.449	-0.116	-0.161	-0.141	-0.147	-0.145	56.16
680405	30.00	460	-0.301	-0.301	-0.115	-0.135	-0.128	-0.130	-0.129	50.78
680405	35.00	460	-0.215	-0.215	-0.110	-0.119	-0.116	-0.117	-0.117	46.34
680405	40.00	460	-0.165	-0.165	-0.103	-0.108	-0.107	-0.107	-0.107	42.70
680405	45.00	460	-0.134	-0.134	-0.097	-0.100	-0.099	-0.099	-0.099	39.72
680405	50.00	460	-0.114	-0.114	-0.092	-0.093	-0.093	-0.093	-0.093	37.30
680405	55.00	460	-0.101	-0.101	-0.087	-0.088	-0.088	-0.088	-0.088	35.32
680405	60.00	460	-0.092	-0.092	-0.084	-0.084	-0.084	-0.084	-0.084	33.73
680405	65.00	460	-0.085	-0.085	-0.081	-0.081	-0.081	-0.081	-0.081	32.46
680405	70.00	460	-0.081	-0.081	-0.078	-0.078	-0.078	-0.078	-0.078	31.48
680405	75.00	460	-0.078	-0.078	-0.076	-0.076	-0.076	-0.076	-0.076	30.74
680405	80.00	460	-0.076	-0.076	-0.075	-0.075	-0.075	-0.075	-0.075	30.23
680405	89.00	460	-0.074	-0.074	-0.074	-0.074	-0.074	-0.074	-0.074	29.84
680405	AVG		-1.080	-1.080	-0.065	-0.346	-0.079	-0.254	-0.102	
680405	RMS		1.563	1.563	0.076	0.661	0.094	0.325	0.110	

Table D-3. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 407 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680410	0.00	407	-3.598	-3.598	0.645	-1.235	0.857	-1.097	1.048	100.92
680410	1.00	407	-3.573	-3.573	0.645	-1.219	0.850	-1.079	1.034	100.78
680410	2.00	407	-3.504	-3.504	0.638	-1.179	0.824	-1.030	0.986	100.38
680410	3.00	407	-3.395	-3.395	0.625	-1.118	0.781	-0.956	0.911	99.72
680410	4.00	407	-3.251	-3.251	0.606	-1.040	0.725	-0.864	0.816	98.82
680410	5.00	407	-3.079	-3.079	0.582	-0.949	0.659	-0.759	0.711	97.71
680410	7.50	407	-2.574	-2.574	0.504	-0.700	0.482	-0.492	0.451	94.16
680410	10.00	407	-2.045	-2.045	0.414	-0.468	0.326	-0.273	0.258	89.84
680410	12.50	407	-1.564	-1.564	0.327	-0.286	0.212	-0.128	0.144	85.16
680410	15.00	407	-1.165	-1.165	0.252	-0.159	0.140	-0.043	0.088	80.41
680410	20.00	407	-0.617	-0.617	0.149	-0.027	0.074	0.025	0.053	71.46
680410	25.00	407	-0.315	-0.315	0.094	0.020	0.054	0.041	0.046	63.74
680410	30.00	407	-0.156	-0.156	0.067	0.034	0.046	0.042	0.044	57.34
680410	35.00	407	-0.072	-0.072	0.052	0.037	0.042	0.041	0.041	52.13
680410	40.00	407	-0.028	-0.028	0.044	0.037	0.039	0.038	0.038	47.89
680410	45.00	407	-0.003	-0.003	0.039	0.035	0.036	0.036	0.036	44.46
680410	50.00	407	0.010	0.010	0.036	0.034	0.034	0.034	0.034	41.67
680410	55.00	407	0.018	0.018	0.033	0.032	0.032	0.032	0.032	39.42
680410	60.00	407	0.022	0.022	0.031	0.031	0.031	0.031	0.031	37.61
680410	65.00	407	0.025	0.025	0.030	0.030	0.030	0.030	0.030	36.17
680410	70.00	407	0.026	0.026	0.029	0.029	0.029	0.029	0.029	35.05
680410	75.00	407	0.027	0.027	0.028	0.028	0.028	0.028	0.028	34.22
680410	80.00	407	0.027	0.027	0.028	0.028	0.028	0.028	0.028	33.64
680410	89.00	407	0.027	0.027	0.027	0.027	0.027	0.027	0.027	33.19
680410	AVG		-1.198	-1.198	0.247	-0.332	0.266	-0.261	0.289	
680410	RMS		1.879	1.879	0.352	0.840	0.415	0.501	0.477	

Table D-4. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 444 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680412	0.00	444	-2.775	-2.775	0.311	-0.774	0.334	-0.522	0.296	84.08
680412	1.00	444	-2.757	-2.757	0.311	-0.765	0.332	-0.514	0.292	83.98
680412	2.00	444	-2.710	-2.710	0.308	-0.744	0.321	-0.494	0.278	83.67
680412	3.00	444	-2.633	-2.633	0.302	-0.710	0.304	-0.464	0.256	83.18
680412	4.00	444	-2.533	-2.533	0.293	-0.667	0.281	-0.426	0.227	82.50
680412	5.00	444	-2.411	-2.411	0.282	-0.616	0.254	-0.384	0.194	81.66
680412	7.50	444	-2.050	-2.050	0.244	-0.475	0.179	-0.271	0.112	78.94
680412	10.00	444	-1.664	-1.664	0.197	-0.341	0.111	-0.176	0.049	75.60
680412	12.50	444	-1.305	-1.305	0.151	-0.232	0.060	-0.108	0.012	71.94
680412	15.00	444	-0.999	-0.999	0.110	-0.153	0.027	-0.067	-0.006	68.18
680412	20.00	444	-0.566	-0.566	0.051	-0.066	-0.002	-0.029	-0.015	60.98
680412	25.00	444	-0.319	-0.319	0.019	-0.032	-0.009	-0.017	-0.014	54.66
680412	30.00	444	-0.183	-0.183	0.004	-0.019	-0.010	-0.013	-0.012	49.36
680412	35.00	444	-0.109	-0.109	-0.002	-0.013	-0.010	-0.011	-0.011	44.99
680412	40.00	444	-0.067	-0.067	-0.006	-0.011	-0.010	-0.010	-0.010	41.42
680412	45.00	444	-0.043	-0.043	-0.007	-0.010	-0.009	-0.009	-0.009	38.51
680412	50.00	444	-0.029	-0.029	-0.007	-0.009	-0.009	-0.009	-0.009	36.14
680412	55.00	444	-0.021	-0.021	-0.008	-0.008	-0.008	-0.008	-0.008	34.21
680412	60.00	444	-0.016	-0.016	-0.008	-0.008	-0.008	-0.008	-0.008	32.66
680412	65.00	444	-0.012	-0.012	-0.008	-0.008	-0.008	-0.008	-0.008	31.43
680412	70.00	444	-0.010	-0.010	-0.007	-0.008	-0.008	-0.008	-0.008	30.47
680412	75.00	444	-0.009	-0.009	-0.007	-0.007	-0.007	-0.007	-0.007	29.75
680412	80.00	444	-0.008	-0.008	-0.007	-0.007	-0.007	-0.007	-0.007	29.26
680412	89.00	444	-0.007	-0.007	-0.007	-0.007	-0.007	-0.007	-0.007	28.87
680412	AVG		-0.968	-0.968	0.105	-0.237	0.087	-0.149	0.066	
680412	RMS		1.471	1.471	0.169	0.540	0.160	0.246	0.132	

Table D-5. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 426 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680417	0.00	426	-3.431	-3.431	0.453	-1.096	0.580	-0.855	0.625	96.20
680417	1.00	426	-3.408	-3.408	0.453	-1.083	0.575	-0.841	0.617	96.07
680417	2.00	426	-3.346	-3.346	0.449	-1.051	0.557	-0.807	0.588	95.71
680417	3.00	426	-3.247	-3.247	0.440	-1.001	0.527	-0.755	0.541	95.11
680417	4.00	426	-3.117	-3.117	0.428	-0.937	0.488	-0.689	0.482	94.29
680417	5.00	426	-2.960	-2.960	0.411	-0.862	0.441	-0.614	0.416	93.28
680417	7.50	426	-2.499	-2.499	0.355	-0.655	0.315	-0.421	0.252	90.02
680417	10.00	426	-2.010	-2.010	0.289	-0.459	0.201	-0.260	0.128	86.05
680417	12.50	426	-1.561	-1.561	0.223	-0.304	0.117	-0.149	0.054	81.72
680417	15.00	426	-1.183	-1.183	0.165	-0.193	0.064	-0.082	0.018	77.31
680417	20.00	426	-0.656	-0.656	0.083	-0.073	0.016	-0.024	-0.002	68.91
680417	25.00	426	-0.360	-0.360	0.040	-0.027	0.003	-0.008	-0.003	61.62
680417	30.00	426	-0.201	-0.201	0.019	-0.010	0.000	-0.003	-0.002	55.54
680417	35.00	426	-0.114	-0.114	0.010	-0.004	-0.000	-0.001	-0.001	50.56
680417	40.00	426	-0.067	-0.067	0.005	-0.002	-0.000	-0.000	-0.000	46.50
680417	45.00	426	-0.040	-0.040	0.002	-0.001	-0.000	-0.000	-0.000	43.20
680417	50.00	426	-0.024	-0.024	0.001	-0.001	-0.000	-0.000	-0.000	40.51
680417	55.00	426	-0.015	-0.015	0.001	-0.000	-0.000	-0.000	-0.000	38.34
680417	60.00	426	-0.009	-0.009	0.000	-0.000	-0.000	-0.000	-0.000	36.59
680417	65.00	426	-0.006	-0.006	-0.000	-0.000	-0.000	-0.000	-0.000	35.20
680417	70.00	426	-0.003	-0.003	-0.000	-0.000	-0.000	-0.000	-0.000	34.12
680417	75.00	426	-0.002	-0.002	-0.000	-0.000	-0.000	-0.000	-0.000	33.31
680417	80.00	426	-0.001	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	32.75
680417	89.00	426	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	32.32
680417	AVG		-1.178	-1.178	0.159	-0.323	0.162	-0.230	0.155	
680417	RMS		1.807	1.807	0.246	0.757	0.277	0.397	0.281	

Table D-6. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 483 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680524	0.00	483	-5.032	-5.032	-0.954	-1.988	-0.953	-1.579	-1.048	131.44
680524	1.00	483	-5.006	-5.006	-0.949	-1.976	-0.950	-1.569	-1.045	131.29
680524	2.00	483	-4.936	-4.936	-0.941	-1.946	-0.948	-1.547	-1.043	130.85
680524	3.00	483	-4.826	-4.826	-0.931	-1.901	-0.947	-1.514	-1.042	130.14
680524	4.00	483	-4.679	-4.679	-0.919	-1.843	-0.946	-1.472	-1.040	129.17
680524	5.00	483	-4.502	-4.502	-0.905	-1.774	-0.945	-1.423	-1.037	127.96
680524	7.50	483	-3.967	-3.967	-0.868	-1.575	-0.939	-1.285	-1.021	124.03
680524	10.00	483	-3.383	-3.383	-0.833	-1.372	-0.922	-1.149	-0.989	119.17
680524	12.50	483	-2.824	-2.824	-0.799	-1.191	-0.892	-1.030	-0.941	113.77
680524	15.00	483	-2.333	-2.333	-0.767	-1.042	-0.852	-0.931	-0.885	108.18
680524	20.00	483	-1.600	-1.600	-0.704	-0.831	-0.760	-0.784	-0.773	97.31
680524	25.00	483	-1.143	-1.143	-0.641	-0.699	-0.673	-0.680	-0.677	87.62
680524	30.00	483	-0.866	-0.866	-0.583	-0.609	-0.599	-0.602	-0.601	79.40
680524	35.00	483	-0.694	-0.694	-0.532	-0.544	-0.541	-0.541	-0.541	72.56
680524	40.00	483	-0.585	-0.585	-0.489	-0.495	-0.494	-0.494	-0.494	66.94
680524	45.00	483	-0.511	-0.511	-0.454	-0.457	-0.456	-0.456	-0.456	62.32
680524	50.00	483	-0.460	-0.460	-0.425	-0.427	-0.426	-0.426	-0.426	58.55
680524	55.00	483	-0.423	-0.423	-0.402	-0.402	-0.402	-0.402	-0.402	55.49
680524	60.00	483	-0.395	-0.395	-0.383	-0.383	-0.383	-0.383	-0.383	53.01
680524	65.00	483	-0.375	-0.375	-0.368	-0.368	-0.368	-0.368	-0.368	51.03
680524	70.00	483	-0.360	-0.360	-0.356	-0.356	-0.356	-0.356	-0.356	49.49
680524	75.00	483	-0.350	-0.350	-0.347	-0.348	-0.348	-0.348	-0.348	48.34
680524	80.00	483	-0.342	-0.342	-0.342	-0.342	-0.342	-0.342	-0.342	47.54
680524	89.00	483	-0.337	-0.337	-0.337	-0.337	-0.337	-0.337	-0.337	46.92
680524	AVG		-2.080	-2.080	-0.635	-0.967	-0.657	-0.834	-0.691	
680524	RMS		2.796	2.796	0.676	1.635	0.704	0.958	0.749	

Table D-7. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 442 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680525	0.00	442	-2.785	-2.785	0.085	-0.934	0.097	-0.707	0.063	79.80
680525	1.00	442	-2.768	-2.768	0.085	-0.926	0.095	-0.699	0.060	79.70
680525	2.00	442	-2.723	-2.723	0.083	-0.905	0.086	-0.680	0.047	79.41
680525	3.00	442	-2.651	-2.651	0.079	-0.872	0.071	-0.650	0.027	78.94
680525	4.00	442	-2.555	-2.555	0.072	-0.830	0.051	-0.612	0.001	78.29
680525	5.00	442	-2.440	-2.440	0.063	-0.781	0.028	-0.569	-0.027	77.49
680525	7.50	442	-2.098	-2.098	0.032	-0.642	-0.035	-0.456	-0.097	74.90
680525	10.00	442	-1.729	-1.729	-0.003	-0.507	-0.089	-0.356	-0.146	71.73
680525	12.50	442	-1.386	-1.386	-0.037	-0.395	-0.125	-0.282	-0.170	68.24
680525	15.00	442	-1.092	-1.092	-0.066	-0.312	-0.145	-0.232	-0.176	64.66
680525	20.00	442	-0.671	-0.671	-0.102	-0.211	-0.152	-0.176	-0.164	57.81
680525	25.00	442	-0.425	-0.425	-0.114	-0.161	-0.141	-0.148	-0.145	51.80
680525	30.00	442	-0.286	-0.286	-0.114	-0.135	-0.128	-0.129	-0.129	46.77
680525	35.00	442	-0.206	-0.206	-0.109	-0.119	-0.116	-0.116	-0.116	42.62
680525	40.00	442	-0.159	-0.159	-0.102	-0.107	-0.106	-0.106	-0.106	39.24
680525	45.00	442	-0.130	-0.130	-0.096	-0.099	-0.098	-0.098	-0.098	36.47
680525	50.00	442	-0.111	-0.111	-0.091	-0.092	-0.092	-0.092	-0.092	34.23
680525	55.00	442	-0.099	-0.099	-0.086	-0.087	-0.087	-0.087	-0.087	32.40
680525	60.00	442	-0.090	-0.090	-0.083	-0.083	-0.083	-0.083	-0.083	30.93
680525	65.00	442	-0.084	-0.084	-0.079	-0.080	-0.080	-0.080	-0.080	29.76
680525	70.00	442	-0.080	-0.080	-0.077	-0.077	-0.077	-0.077	-0.077	28.85
680525	75.00	442	-0.077	-0.077	-0.075	-0.075	-0.075	-0.075	-0.075	28.17
680525	80.00	442	-0.075	-0.075	-0.074	-0.074	-0.074	-0.074	-0.074	27.70
680525	89.00	442	-0.073	-0.073	-0.073	-0.073	-0.073	-0.073	-0.073	27.34
680525	AVG		-1.033	-1.033	-0.037	-0.357	-0.056	-0.277	-0.076	
680525	RMS		1.496	1.496	0.082	0.690	0.097	0.365	0.103	

Table D-8. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 467 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680529	0.00	467	-2.742	-2.742	0.111	-0.774	0.060	-0.516	-0.013	93.93
680529	1.00	467	-2.726	-2.726	0.112	-0.767	0.059	-0.510	-0.014	93.82
680529	2.00	467	-2.683	-2.683	0.110	-0.749	0.054	-0.496	-0.020	93.50
680529	3.00	467	-2.614	-2.614	0.107	-0.722	0.045	-0.475	-0.029	92.99
680529	4.00	467	-2.522	-2.522	0.102	-0.687	0.032	-0.448	-0.042	92.28
680529	5.00	467	-2.411	-2.411	0.095	-0.645	0.018	-0.417	-0.056	91.40
680529	7.50	467	-2.080	-2.080	0.070	-0.528	-0.023	-0.335	-0.091	88.54
680529	10.00	467	-1.720	-1.720	0.040	-0.413	-0.059	-0.262	-0.115	85.01
680529	12.50	467	-1.379	-1.379	0.009	-0.317	-0.084	-0.206	-0.126	81.10
680529	15.00	467	-1.085	-1.085	-0.017	-0.244	-0.098	-0.167	-0.126	77.05
680529	20.00	467	-0.657	-0.657	-0.053	-0.156	-0.102	-0.123	-0.113	69.20
680529	25.00	467	-0.403	-0.403	-0.068	-0.114	-0.095	-0.101	-0.098	62.22
680529	30.00	467	-0.259	-0.259	-0.071	-0.092	-0.085	-0.087	-0.087	56.32
680529	35.00	467	-0.177	-0.177	-0.070	-0.080	-0.077	-0.078	-0.077	51.43
680529	40.00	467	-0.129	-0.129	-0.067	-0.072	-0.070	-0.071	-0.071	47.41
680529	45.00	467	-0.100	-0.100	-0.063	-0.066	-0.065	-0.065	-0.065	44.12
680529	50.00	467	-0.082	-0.082	-0.060	-0.061	-0.061	-0.061	-0.061	41.43
680529	55.00	467	-0.071	-0.071	-0.057	-0.058	-0.057	-0.057	-0.057	39.25
680529	60.00	467	-0.063	-0.063	-0.054	-0.055	-0.055	-0.055	-0.055	37.49
680529	65.00	467	-0.057	-0.057	-0.052	-0.053	-0.053	-0.053	-0.053	36.08
680529	70.00	467	-0.054	-0.054	-0.051	-0.051	-0.051	-0.051	-0.051	34.99
680529	75.00	467	-0.051	-0.051	-0.050	-0.050	-0.050	-0.050	-0.050	34.17
680529	80.00	467	-0.049	-0.049	-0.049	-0.049	-0.049	-0.049	-0.049	33.61
680529	89.00	467	-0.048	-0.048	-0.048	-0.048	-0.048	-0.048	-0.048	33.17
680529	AVG		-1.007	-1.007	-0.003	-0.285	-0.038	-0.199	-0.065	
680529	RMS		1.476	1.476	0.071	0.567	0.064	0.266	0.073	

Table D-9. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 445 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680530	0.00	445	-2.771	-2.771	0.265	-0.781	0.277	-0.528	0.233	83.39
680530	1.00	445	-2.754	-2.754	0.266	-0.773	0.275	-0.521	0.230	83.29
680530	2.00	445	-2.707	-2.707	0.263	-0.752	0.265	-0.502	0.218	82.99
680530	3.00	445	-2.632	-2.632	0.258	-0.719	0.250	-0.473	0.198	82.50
680530	4.00	445	-2.532	-2.532	0.250	-0.677	0.229	-0.437	0.172	81.83
680530	5.00	445	-2.411	-2.411	0.240	-0.627	0.204	-0.396	0.143	80.99
680530	7.50	445	-2.054	-2.054	0.205	-0.489	0.136	-0.287	0.070	78.30
680530	10.00	445	-1.670	-1.670	0.163	-0.357	0.075	-0.195	0.015	75.00
680530	12.50	445	-1.313	-1.313	0.121	-0.249	0.031	-0.128	-0.016	71.37
680530	15.00	445	-1.009	-1.009	0.084	-0.171	0.002	-0.086	-0.030	67.65
680530	20.00	445	-0.577	-0.577	0.031	-0.082	-0.021	-0.046	-0.033	60.51
680530	25.00	445	-0.330	-0.330	0.004	-0.046	-0.024	-0.031	-0.028	54.24
680530	30.00	445	-0.193	-0.193	-0.008	-0.031	-0.023	-0.025	-0.024	48.98
680530	35.00	445	-0.118	-0.118	-0.013	-0.024	-0.021	-0.021	-0.021	44.65
680530	40.00	445	-0.076	-0.076	-0.015	-0.020	-0.019	-0.019	-0.019	41.11
680530	45.00	445	-0.051	-0.051	-0.015	-0.018	-0.017	-0.017	-0.017	38.22
680530	50.00	445	-0.037	-0.037	-0.015	-0.016	-0.016	-0.016	-0.016	35.87
680530	55.00	445	-0.028	-0.028	-0.015	-0.015	-0.015	-0.015	-0.015	33.96
680530	60.00	445	-0.022	-0.022	-0.014	-0.015	-0.015	-0.015	-0.015	32.42
680530	65.00	445	-0.019	-0.019	-0.014	-0.014	-0.014	-0.014	-0.014	31.20
680530	70.00	445	-0.016	-0.016	-0.013	-0.014	-0.014	-0.014	-0.014	30.25
680530	75.00	445	-0.015	-0.015	-0.013	-0.013	-0.013	-0.013	-0.013	29.54
680530	80.00	445	-0.014	-0.014	-0.013	-0.013	-0.013	-0.013	-0.013	29.04
680530	89.00	445	-0.013	-0.013	-0.013	-0.013	-0.013	-0.013	-0.013	28.66
680530	AVG		-0.973	-0.973	0.083	-0.247	0.063	-0.159	0.041	
680530	RMS		1.472	1.472	0.143	0.549	0.130	0.252	0.103	

Table D-10. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 449 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680604	0.00	449	-2.381	-2.381	0.148	-0.703	0.123	-0.498	0.076	79.80
680604	1.00	449	-2.367	-2.367	0.148	-0.696	0.121	-0.492	0.074	79.70
680604	2.00	449	-2.327	-2.327	0.146	-0.679	0.115	-0.477	0.066	79.42
680604	3.00	449	-2.265	-2.265	0.142	-0.653	0.104	-0.454	0.052	78.96
680604	4.00	449	-2.182	-2.182	0.135	-0.619	0.089	-0.425	0.034	78.34
680604	5.00	449	-2.083	-2.083	0.127	-0.579	0.071	-0.392	0.014	77.55
680604	7.50	449	-1.786	-1.786	0.099	-0.467	0.023	-0.305	-0.034	75.03
680604	10.00	449	-1.465	-1.465	0.067	-0.358	-0.019	-0.228	-0.069	71.93
680604	12.50	449	-1.165	-1.165	0.035	-0.268	-0.048	-0.172	-0.086	68.50
680604	15.00	449	-0.908	-0.908	0.008	-0.201	-0.064	-0.133	-0.090	64.98
680604	20.00	449	-0.539	-0.539	-0.028	-0.121	-0.072	-0.092	-0.082	58.19
680604	25.00	449	-0.324	-0.324	-0.043	-0.084	-0.067	-0.073	-0.071	52.21
680604	30.00	449	-0.204	-0.204	-0.048	-0.066	-0.060	-0.062	-0.061	47.17
680604	35.00	449	-0.136	-0.136	-0.048	-0.056	-0.054	-0.054	-0.054	43.02
680604	40.00	449	-0.097	-0.097	-0.046	-0.050	-0.049	-0.049	-0.049	39.62
680604	45.00	449	-0.074	-0.074	-0.043	-0.045	-0.045	-0.045	-0.045	36.84
680604	50.00	449	-0.059	-0.059	-0.041	-0.042	-0.042	-0.042	-0.042	34.58
680604	55.00	449	-0.050	-0.050	-0.039	-0.040	-0.039	-0.039	-0.039	32.74
680604	60.00	449	-0.044	-0.044	-0.037	-0.038	-0.037	-0.037	-0.037	31.26
680604	65.00	449	-0.040	-0.040	-0.036	-0.036	-0.036	-0.036	-0.036	30.08
680604	70.00	449	-0.037	-0.037	-0.035	-0.035	-0.035	-0.035	-0.035	29.17
680604	75.00	449	-0.035	-0.035	-0.034	-0.034	-0.034	-0.034	-0.034	28.48
680604	80.00	449	-0.034	-0.034	-0.033	-0.033	-0.033	-0.033	-0.033	28.01
680604	89.00	449	-0.033	-0.033	-0.033	-0.033	-0.033	-0.033	-0.033	27.64
680604	AVG		-0.860	-0.860	0.021	-0.247	-0.005	-0.177	-0.026	
680604	RMS		1.273	1.273	0.081	0.507	0.066	0.249	0.056	

Table D-11. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 427 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680605	0.00	427	-3.385	-3.385	0.389	-0.974	0.441	-0.715	0.439	104.35
680605	1.00	427	-3.362	-3.362	0.390	-0.962	0.438	-0.703	0.434	104.22
680605	2.00	427	-3.301	-3.301	0.387	-0.933	0.424	-0.675	0.413	103.82
680605	3.00	427	-3.204	-3.204	0.379	-0.889	0.401	-0.632	0.379	103.18
680605	4.00	427	-3.076	-3.076	0.369	-0.832	0.370	-0.578	0.336	102.31
680605	5.00	427	-2.923	-2.923	0.354	-0.766	0.334	-0.517	0.287	101.23
680605	7.50	427	-2.469	-2.469	0.306	-0.584	0.235	-0.358	0.167	97.74
680605	10.00	427	-1.988	-1.988	0.248	-0.412	0.147	-0.225	0.077	93.47
680605	12.50	427	-1.544	-1.544	0.190	-0.274	0.082	-0.133	0.026	88.81
680605	15.00	427	-1.172	-1.172	0.140	-0.175	0.042	-0.076	0.002	84.05
680605	20.00	427	-0.650	-0.650	0.070	-0.068	0.008	-0.024	-0.007	74.97
680605	25.00	427	-0.356	-0.356	0.034	-0.025	0.001	-0.008	-0.005	67.06
680605	30.00	427	-0.197	-0.197	0.017	-0.009	-0.000	-0.003	-0.002	60.46
680605	35.00	427	-0.112	-0.112	0.009	-0.003	0.000	-0.000	-0.000	55.04
680605	40.00	427	-0.064	-0.064	0.005	-0.001	0.001	0.001	0.001	50.63
680605	45.00	427	-0.038	-0.038	0.004	0.001	0.001	0.001	0.001	47.04
680605	50.00	427	-0.022	-0.022	0.003	0.001	0.001	0.001	0.001	44.12
680605	55.00	427	-0.013	-0.013	0.002	0.001	0.001	0.001	0.001	41.75
680605	60.00	427	-0.007	-0.007	0.002	0.001	0.002	0.001	0.001	39.85
680605	65.00	427	-0.004	-0.004	0.002	0.001	0.002	0.002	0.002	38.33
680605	70.00	427	-0.001	-0.001	0.002	0.001	0.002	0.002	0.002	37.16
680605	75.00	427	0.000	0.000	0.002	0.002	0.002	0.002	0.002	36.28
680605	80.00	427	0.001	0.001	0.002	0.001	0.001	0.001	0.001	35.67
680605	89.00	427	0.001	0.001	0.001	0.001	0.001	0.001	0.001	35.20
680605	AVG		-1.162	-1.162	0.138	-0.287	0.122	-0.193	0.107	
680605	RMS		1.783	1.783	0.212	0.673	0.210	0.333	0.196	

Table D-12. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 389 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680611	0.00	389	-7.072	-7.072	0.148	-2.858	1.714	-2.576	2.902	87.22
680611	1.00	389	-7.015	-7.015	0.159	-2.820	1.700	-2.530	2.859	87.07
680611	2.00	389	-6.854	-6.854	0.185	-2.717	1.653	-2.407	2.727	86.62
680611	3.00	389	-6.601	-6.601	0.222	-2.557	1.576	-2.217	2.520	85.89
680611	4.00	389	-6.270	-6.270	0.264	-2.353	1.474	-1.979	2.261	84.91
680611	5.00	389	-5.881	-5.881	0.306	-2.120	1.353	-1.714	1.973	83.72
680611	7.50	389	-4.777	-4.777	0.387	-1.500	1.018	-1.055	1.267	79.99
680611	10.00	389	-3.679	-3.679	0.411	-0.955	0.709	-0.541	0.738	75.66
680611	12.50	389	-2.732	-2.732	0.385	-0.552	0.476	-0.219	0.420	71.14
680611	15.00	389	-1.981	-1.981	0.334	-0.287	0.320	-0.046	0.254	66.72
680611	20.00	389	-1.003	-1.003	0.228	-0.032	0.168	0.073	0.134	58.68
680611	25.00	389	-0.494	-0.494	0.156	0.048	0.113	0.089	0.101	52.00
680611	30.00	389	-0.236	-0.236	0.114	0.068	0.090	0.083	0.086	46.59
680611	35.00	389	-0.104	-0.104	0.090	0.069	0.077	0.075	0.076	42.24
680611	40.00	389	-0.035	-0.035	0.076	0.066	0.069	0.069	0.069	38.74
680611	45.00	389	0.001	0.001	0.067	0.062	0.063	0.063	0.063	35.92
680611	50.00	389	0.021	0.021	0.060	0.058	0.058	0.058	0.058	33.64
680611	55.00	389	0.032	0.032	0.056	0.055	0.055	0.055	0.055	31.80
680611	60.00	389	0.038	0.038	0.053	0.052	0.052	0.052	0.052	30.33
680611	65.00	389	0.042	0.042	0.050	0.050	0.050	0.050	0.050	29.16
680611	70.00	389	0.044	0.044	0.048	0.048	0.048	0.048	0.048	28.25
680611	75.00	389	0.044	0.044	0.047	0.047	0.047	0.047	0.047	27.57
680611	80.00	389	0.045	0.045	0.046	0.046	0.046	0.046	0.046	27.10
680611	89.00	389	0.045	0.045	0.045	0.045	0.045	0.045	0.045	26.74
680611	AVG		-2.268	-2.268	0.164	-0.752	0.541	-0.601	0.785	
680611	RMS		3.610	3.610	0.204	1.908	0.842	1.157	1.320	

Table D-13. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 409 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680613	0.00	409	-7.725	-7.725	-0.258	-2.860	1.149	-2.102	1.851	97.06
680613	1.00	409	-7.665	-7.665	-0.243	-2.824	1.143	-2.066	1.827	96.90
680613	2.00	409	-7.497	-7.497	-0.209	-2.725	1.116	-1.970	1.747	96.43
680613	3.00	409	-7.232	-7.232	-0.158	-2.572	1.070	-1.822	1.621	95.66
680613	4.00	409	-6.886	-6.886	-0.098	-2.378	1.007	-1.637	1.461	94.62
680613	5.00	409	-6.478	-6.478	-0.033	-2.154	0.931	-1.431	1.281	93.36
680613	7.50	409	-5.312	-5.312	0.110	-1.559	0.710	-0.914	0.829	89.40
680613	10.00	409	-4.141	-4.141	0.193	-1.031	0.494	-0.506	0.478	84.76
680613	12.50	409	-3.120	-3.120	0.215	-0.636	0.323	-0.245	0.259	79.89
680613	15.00	409	-2.303	-2.303	0.199	-0.373	0.203	-0.102	0.141	75.09
680613	20.00	409	-1.222	-1.222	0.131	-0.114	0.082	-0.000	0.051	66.31
680613	25.00	409	-0.649	-0.649	0.076	-0.028	0.037	0.016	0.026	58.93
680613	30.00	409	-0.352	-0.352	0.042	-0.003	0.020	0.014	0.016	52.91
680613	35.00	409	-0.197	-0.197	0.024	0.003	0.012	0.010	0.010	48.04
680613	40.00	409	-0.113	-0.113	0.014	0.004	0.007	0.006	0.007	44.11
680613	45.00	409	-0.067	-0.067	0.008	0.003	0.004	0.004	0.004	40.93
680613	50.00	409	-0.040	-0.040	0.004	0.002	0.002	0.002	0.002	38.36
680613	55.00	409	-0.025	-0.025	0.002	0.001	0.001	0.001	0.001	36.28
680613	60.00	409	-0.015	-0.015	0.001	0.000	0.000	0.000	0.000	34.61
680613	65.00	409	-0.010	-0.010	-0.000	-0.001	-0.000	-0.000	-0.000	33.28
680613	70.00	409	-0.006	-0.006	-0.001	-0.001	-0.001	-0.001	-0.001	32.25
680613	75.00	409	-0.004	-0.004	-0.001	-0.001	-0.001	-0.001	-0.001	31.49
680613	80.00	409	-0.003	-0.003	-0.001	-0.001	-0.001	-0.001	-0.001	30.96
680613	89.00	409	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002	30.54
680613	AVG		-2.544	-2.544	0.001	-0.802	0.346	-0.531	0.484	
680613	RMS		3.973	3.973	0.123	1.927	0.570	0.953	0.847	

Table D-14. Ionospheric Range Error Difference, DIFF(N), Between Moment Expansion to Order N About a Centroid of 392 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680618	0.00	392	-9.524	-9.524	0.353	-3.858	2.207	-3.476	3.600	135.26
680618	1.00	392	-9.446	-9.446	0.368	-3.806	2.191	-3.414	3.548	135.04
680618	2.00	392	-9.235	-9.235	0.397	-3.670	2.130	-3.251	3.383	134.38
680618	3.00	392	-8.902	-8.902	0.434	-3.461	2.027	-3.003	3.123	133.31
680618	4.00	392	-8.469	-8.469	0.475	-3.194	1.889	-2.693	2.796	131.86
680618	5.00	392	-7.959	-7.959	0.514	-2.890	1.727	-2.347	2.431	130.09
680618	7.50	392	-6.507	-6.507	0.575	-2.079	1.277	-1.483	1.534	124.56
680618	10.00	392	-5.052	-5.052	0.564	-1.360	0.863	-0.807	0.861	118.06
680618	12.50	392	-3.788	-3.788	0.498	-0.824	0.552	-0.379	0.458	111.22
680618	15.00	392	-2.778	-2.778	0.410	-0.468	0.347	-0.145	0.249	104.47
680618	20.00	392	-1.451	-1.451	0.249	-0.120	0.149	0.022	0.102	92.11
680618	25.00	392	-0.752	-0.752	0.147	-0.007	0.081	0.049	0.065	81.74
680618	30.00	392	-0.394	-0.394	0.092	0.025	0.056	0.047	0.050	73.31
680618	35.00	392	-0.208	-0.208	0.062	0.032	0.043	0.041	0.041	66.50
680618	40.00	392	-0.109	-0.109	0.046	0.031	0.036	0.035	0.035	61.01
680618	45.00	392	-0.055	-0.055	0.036	0.029	0.031	0.031	0.031	56.58
680618	50.00	392	-0.024	-0.024	0.030	0.027	0.027	0.027	0.027	53.00
680618	55.00	392	-0.006	-0.006	0.026	0.024	0.025	0.025	0.025	50.11
680618	60.00	392	0.004	0.004	0.024	0.022	0.023	0.023	0.023	47.79
680618	65.00	392	0.010	0.010	0.022	0.021	0.021	0.021	0.021	45.95
680618	70.00	392	0.014	0.014	0.020	0.020	0.020	0.020	0.020	44.52
680618	75.00	392	0.016	0.016	0.019	0.019	0.019	0.019	0.019	43.45
680618	80.00	392	0.017	0.017	0.019	0.019	0.019	0.019	0.019	42.72
680618	89.00	392	0.018	0.018	0.018	0.018	0.018	0.018	0.018	42.15
680618	AVG		-3.107	-3.107	0.225	-1.060	0.657	-0.859	0.937	
680618	RMS		4.883	4.883	0.307	2.590	1.074	1.569	1.631	

Table D-15. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 400 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680621	0.00	400	-1.731	-1.731	0.452	-0.539	0.547	-0.498	0.670	58.76
680621	1.00	400	-1.718	-1.718	0.451	-0.532	0.543	-0.489	0.661	58.69
680621	2.00	400	-1.684	-1.684	0.445	-0.513	0.527	-0.464	0.633	58.45
680621	3.00	400	-1.630	-1.630	0.435	-0.483	0.502	-0.425	0.589	58.07
680621	4.00	400	-1.559	-1.559	0.421	-0.445	0.469	-0.377	0.533	57.55
680621	5.00	400	-1.474	-1.474	0.404	-0.400	0.432	-0.323	0.472	56.90
680621	7.50	400	-1.223	-1.223	0.352	-0.278	0.330	-0.185	0.321	54.84
680621	10.00	400	-0.961	-0.961	0.293	-0.166	0.239	-0.075	0.208	52.32
680621	12.50	400	-0.723	-0.723	0.238	-0.080	0.173	-0.004	0.140	49.59
680621	15.00	400	-0.527	-0.527	0.191	-0.021	0.130	0.036	0.104	46.82
680621	20.00	400	-0.259	-0.259	0.126	0.036	0.087	0.062	0.076	41.59
680621	25.00	400	-0.114	-0.114	0.091	0.053	0.070	0.063	0.066	37.08
680621	30.00	400	-0.040	-0.040	0.071	0.055	0.060	0.059	0.059	33.35
680621	35.00	400	-0.002	-0.002	0.060	0.052	0.054	0.054	0.054	30.30
680621	40.00	400	0.017	0.017	0.052	0.049	0.050	0.050	0.050	27.83
680621	45.00	400	0.027	0.027	0.047	0.046	0.046	0.046	0.046	25.83
680621	50.00	400	0.031	0.031	0.044	0.043	0.043	0.043	0.043	24.21
680621	55.00	400	0.034	0.034	0.041	0.041	0.041	0.041	0.041	22.90
680621	60.00	400	0.034	0.034	0.039	0.039	0.039	0.039	0.039	21.84
680621	65.00	400	0.035	0.035	0.037	0.037	0.037	0.037	0.037	21.00
680621	70.00	400	0.035	0.035	0.036	0.036	0.036	0.036	0.036	20.35
680621	75.00	400	0.034	0.034	0.035	0.035	0.035	0.035	0.035	19.87
680621	80.00	400	0.034	0.034	0.035	0.034	0.034	0.034	0.034	19.53
680621	89.00	400	0.034	0.034	0.034	0.034	0.034	0.034	0.034	19.27
680621	AVG		-0.555	-0.555	0.185	-0.120	0.190	-0.090	0.207	
680621	RMS		0.898	0.898	0.249	0.361	0.272	0.224	0.313	

Table D-16. Ionospheric Range Error Difference, DIFF (N), Between Moment Expansion to Order N About a Centroid of 410 km and REEK Ray Trace

Date	Elv (deg)	Expn Center Height h_c (km)	DIFF (N) (meters)							REEK Ray Trace (meters)
			N0	N1	N2	N3	N4	N5	N6	
680625	0.00	410	-2.812	-2.812	0.467	-1.026	0.555	-0.925	0.672	86.76
680625	1.00	410	-2.794	-2.794	0.466	-1.014	0.549	-0.912	0.661	86.65
680625	2.00	410	-2.743	-2.743	0.458	-0.985	0.529	-0.876	0.626	86.31
680625	3.00	410	-2.662	-2.662	0.445	-0.940	0.496	-0.821	0.570	85.76
680625	4.00	410	-2.556	-2.556	0.426	-0.882	0.453	-0.752	0.501	85.01
680625	5.00	410	-2.429	-2.429	0.403	-0.814	0.403	-0.674	0.423	84.07
680625	7.50	410	-2.052	-2.052	0.331	-0.628	0.268	-0.472	0.233	81.08
680625	10.00	410	-1.655	-1.655	0.251	-0.452	0.151	-0.306	0.093	77.43
680625	12.50	410	-1.290	-1.290	0.178	-0.312	0.067	-0.192	0.012	73.46
680625	15.00	410	-0.985	-0.985	0.117	-0.212	0.016	-0.124	-0.025	69.42
680625	20.00	410	-0.558	-0.558	0.038	-0.103	-0.026	-0.063	-0.042	61.76
680625	25.00	410	-0.319	-0.319	0.001	-0.059	-0.033	-0.043	-0.039	55.13
680625	30.00	410	-0.189	-0.189	-0.015	-0.041	-0.032	-0.035	-0.034	49.62
680625	35.00	410	-0.118	-0.118	-0.020	-0.032	-0.029	-0.030	-0.030	45.12
680625	40.00	410	-0.078	-0.078	-0.022	-0.028	-0.027	-0.027	-0.027	41.47
680625	45.00	410	-0.055	-0.055	-0.022	-0.025	-0.025	-0.025	-0.025	38.50
680625	50.00	410	-0.041	-0.041	-0.022	-0.023	-0.023	-0.023	-0.023	36.09
680625	55.00	410	-0.033	-0.033	-0.021	-0.022	-0.022	-0.022	-0.022	34.14
680625	60.00	410	-0.027	-0.027	-0.020	-0.021	-0.021	-0.021	-0.021	32.58
680625	65.00	410	-0.024	-0.024	-0.020	-0.020	-0.020	-0.020	-0.020	31.33
680625	70.00	410	-0.021	-0.021	-0.019	-0.019	-0.019	-0.019	-0.019	30.36
680625	75.00	410	-0.020	-0.020	-0.019	-0.019	-0.019	-0.019	-0.019	29.64
680625	80.00	410	-0.019	-0.019	-0.018	-0.018	-0.018	-0.018	-0.018	29.14
680625	89.00	410	-0.018	-0.018	-0.018	-0.018	-0.018	-0.018	-0.018	28.76
680625	AVG		-0.979	-0.979	0.139	-0.321	0.132	-0.268	0.142	
680625	RMS		1.483	1.483	0.242	0.715	0.259	0.434	0.296	

* U.S. GOVERNMENT PRINTING OFFICE: 1972—735-965/441